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(54) Title: METHOD AND APPARATUS FOR MAGNETIC FIELD DETECTION
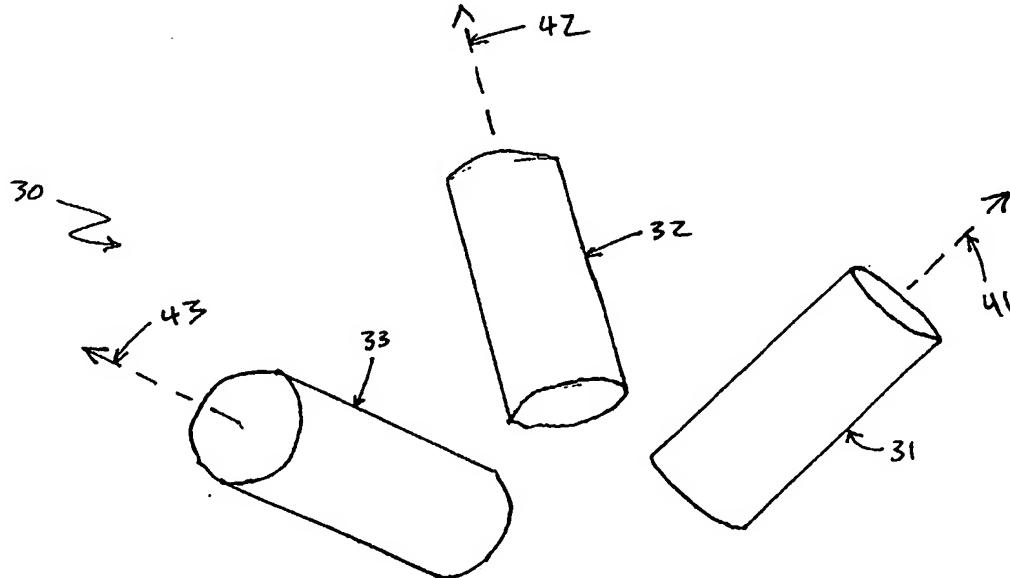
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(57) Abstract: A method and device for field detection are provided, in which one or more gradiometers (31, 32, 33) are positioned in the field and rotated about their axes. Rotation of a single gradiometer allows an output signal of the gradiometer to be analysed in the Fourier domain, which allows particular field components to be obtained, and also separates the field signal from noise, in the frequency domain. Use of three such rotating gradiometers with non-parallel axes allows a complete magnetic field gradient tensor to be obtained with data redundancy, and can reduce crosstalk.

"Method and apparatus for magnetic field detection"

Technical Field

The present invention relates to a method and apparatus for measurement of a magnetic field, and in particular relates to rotating a magnetic sensor relative to the magnetic field to be measured.

Background Art

A gradiometer is a device for measuring the difference between two signals, with an aim to facilitate rejection of common mode noise signals and improved reduction in errors that arise from misorientation of the measuring instrument.

The ability of a gradiometer to measure small magnetic field gradients in the presence of a large background magnetic field is determined by its ability to reject large common mode signals, and the intrinsic noise and drift performance of the field detectors employed. The first of these characteristics is called the common mode rejection ratio (CMRR). In conventional gradiometers the CMRR is determined by the precision with which the field sensors comprising the gradiometer are balanced. That is, the CMRR is determined by the relative sensitivities of the field sensors, and the accuracy to which the field sensors can be positioned in precisely parallel planes. This fundamental nexus between the precision with which conventional gradiometers can be manufactured and the ultimate CMRR performance has until now limited the CMRR to figures of the order of 10^{-4} in practical instruments.

Further, the DC and low frequency accuracy of state of the art gradiometers is limited by the DC and drift performance of the field detectors employed, and is also limited by the low frequency noise performance of the field detectors. Furthermore, the achievable noise performance can be dependent upon the magnitude of the background homogeneous field; being determined by microphonics which arise from vibrations causing randomly varying misalignment of the axes of symmetry of the two field sensors. For example, in the case of magnetic gradiometry two appropriate magnetic sensors are the flux gate and the superconducting quantum interference device (SQUID). Flux gates have considerable drift and noise in comparison to SQUIDs, but state of the art SQUID systems are incapable of providing data about the absolute value of the magnetic field or gradient.

The inability of state of the art SQUIDs to measure the absolute value of the magnetic field or gradient arises for the following reasons. The radio frequency (rf) SQUID is essentially a ring of superconducting material, the ring being interrupted by a Josephson Junction. When the superconducting ring is

5   energised by an inductively coupled resonant rf-oscillator, tunnelling of electrons takes place at the junction and a periodic signal, being a function of flux through the ring, can be detected across the junction. The periodic signal is substantially a triangular waveform, usually having a period (ΔB) in the order of a nanotesla. Therefore, in order to yield a sensitivity in the femtotesla range,

10   the SQUID is operated in a nulling bridge mode, or flux locked loop (FLL) mode. In this mode, magnetic flux is fed back to the SQUID so as to cause the output voltage to remain relatively constant. The feedback voltage, being proportional to the difference between the applied flux and the quiescent flux level, gives a highly accurate measurement of relative magnetic flux. The

15   feedback voltage V can therefore be written as

$$V = M \left(A_{eff}B + u\right) \qquad\qquad (1)$$

where

M is a constant in a specific SQUID system;

A_{eff} is the effective area of the SQUID;

20   B is the applied magnetic field; and

u is the quiescent flux.

However, the quiescent flux u is unknown. Thus, conventional rf-SQUIDs provide only relative measurements of magnetic field, up to an offset (which can usually be held constant for the duration of the measurement

25   procedure) which is an integer multiple of a field value determined by the sensitivity of the SQUID and the fundamental flux quantum. The SQUID, therefore, is not an absolute field detector and state of the art gradiometers cannot provide absolute value measurements of the gradient components.

A further problem arises in tensor gradiometers. The gradient of a field

30   in 3 dimensions is a second rank tensor, and accordingly has 9 components. In the case of magnetic fields Maxwell's equations impose restrictions which imply that only 5 of the gradient components are independent. To determine the gradient tensor of a magnetic field it is therefore necessary to measure all 5 of these components. State of the art tensor gradiometers achieve this by

35   deploying a suite of at least 5 field sensors or planar gradiometers, or a combination of field sensors and planar gradiometers.

However, a problem which arises from the use of a plurality of field sensors or gradiometers to measure the gradient components, is that the sensors or gradiometers distort the field in their neighbourhood. Consequently, each sensor or gradiometer adversely affects the measurement of the field by the other sensors or gradiometers which are deployed to measure other gradient components. Such a "cross-talk" effect limits the accuracy of the gradient measurement.

Any discussion of documents, acts, materials, devices, articles or the like which has been included in the present specification is solely for the purpose of providing a context for the present invention. It is not to be taken as an admission that any or all of these matters form part of the prior art base or were common general knowledge in the field relevant to the present invention as it existed before the priority date of each claim of this application.

Throughout this specification the word "comprise", or variations such as "comprises" or "comprising", will be understood to imply the inclusion of a stated element, integer or step, or group of elements, integers or steps, but not the exclusion of any other element, integer or step, or group of elements, integers or steps.

Throughout this specification, the terms 'superconducting material', 'superconducting device' and the like are used to refer to a material or device which, in a certain state and at a certain temperature, is capable of exhibiting superconductivity. The use of such terms does not imply that the material or device exhibits superconductivity in all states or at all temperatures.

Summary of the Invention

According to a first aspect, the present invention provides a method of field detection comprising:

providing a gradiometer in a field, the gradiometer having at least first and second field vector sensors connected in a differencing arrangement; and

controllably altering a position of the at least first and second field vector sensors relative to the field during operation of the gradiometer.

According to a second aspect the present invention provides a field detection device comprising:

a gradiometer, the gradiometer having at least first and second field vector sensors connected in a differencing arrangement; and

means for controllably altering the position of the at least first and second field vector sensors relative to a field during operation of the gradiometer.

In preferred embodiments of the present invention, the at least first and second field vector sensors of the gradiometer are substantially axially aligned, thus forming an axial gradiometer. That is, the at least first and second field vector sensors are preferably positioned relative to each other such that first and second field vectors sensed by the respective first and second field vector sensors are substantially co-axial. However, the present invention may also be applied in respect of other devices such as planar gradiometers.

In particularly preferred embodiments of the present invention, the position of the at least first and second field vector sensors is controllably altered by rotating the at least first and second field vectors about an axis of rotation. Such rotation may be continuous, as in the case of a motor driving the gradiometer at a substantially constant revolutional velocity about the axis of rotation. Alternatively, the rotation may be piecewise about the axis of rotation, as in the case of manual rotation of the gradiometer through a given angle about the axis of rotation. Measurements may be obtained from the gradiometer before, during, and/or after such a manual rotation.

In preferred embodiments of the invention, the axis of rotation is positioned substantially perpendicular to the co-axial first and second field vectors. The axis of rotation preferably passes substantially between the first and second field vector sensors, and substantially equidistant from the first and second field vector sensors.

In accordance with preferred embodiments of the present invention, the gradiometer is made to rotate so that the direction of the sensitive direction of the field vector sensors or their sensitivities is changed in a known way.

In accordance with preferred embodiments of the invention, the gradiometer may be an axial magnetic gradiometer, and may be formed from a pair of SQUIDs, flux gates or superconducting pick-up loops with their sensitivity vectors lying substantially in a nominal x-y plane. In preferred embodiments, such an axial gradiometer is rotated about a nominal z-axis to form the rotating gradiometer.

While the use of an axial gradiometer may be favoured in implementations of the present invention because of its intrinsic noise immunity, the advantages of rotating a gradiometer are not limited to the axial configuration.

It has been realised by the present inventors that a rotating axial magnetic gradiometer returns information about the value of the g_{xy} component and linear combinations of the values of the g_{xx} and g_{yy} components of a field gradient tensor, in addition to information about the B_x and B_y field components.

5 In preferred embodiments of the invention, such information is retrieved by transforming the voltage waveform from the field vector sensors into the Fourier domain.

Furthermore, it has been realised that it is possible to obtain the complete magnetic gradient tensor by deploying three such rotating

10 gradiometers, configured in such a way that their respective axes of rotation are not parallel.

Accordingly, in a third aspect, the present invention provides a method of obtaining a complete magnetic gradient tensor of a magnetic field, the method comprising:

15 providing at least three axial gradiometers in the magnetic field such that an axis of each axial gradiometer is not parallel to an axis of any other one of the at least three gradiometers; and

controllably altering a position of each of the at least three gradiometers relative to the magnetic field during operation of the gradiometer.

20 Preferably the step of controllably altering comprises rotating each of the at least three axial gradiometers about a respective axis of rotation of that gradiometer. The axis of rotation of each gradiometer is preferably not parallel to an axis of rotation of any other one of the at least three gradiometers.

Similarly, in a fourth aspect, the present invention provides a device for

25 obtaining a complete magnetic gradient tensor of a magnetic field, the device comprising:

at least three axial gradiometers positioned such that an axis of each axial gradiometer is not parallel to an axis of any other one of the at least three gradiometers;

30 means for controllably altering a position of each of the at least three gradiometers relative to the magnetic field during operation of the gradiometer.

Preferably the means for controllably altering comprises means for rotating each of the at least three axial gradiometers about a respective axis of rotation of that gradiometer. The axis of rotation of each gradiometer is

35 preferably not parallel to an axis of rotation of any other one of the at least three gradiometers.

In preferred embodiments of the third and fourth aspects of the invention, the at least three gradiometers are rotated at a substantially constant velocity about their respective axes of rotation. Such embodiments produce data suitable for conversion to the Fourier domain.

5   Embodiments of the third and fourth aspects of the invention effectively provide a single measurement of the true value of each of the three off-diagonal components of the gradient tensor, three measurements of the diagonal components and two measurements of each of the field components. Because more than the minimum of five tensor components are obtained, the use of just three such rotating gradiometers in fact provides a redundant measurement. This redundancy permits a least-squares best fit solution, further improving the quality of data obtained. Any DC offsets such as low frequency drift in flux gates or the fixed offset of a SQUID, for example, can also be determined and monitored to provide information about the operating conditions of the detectors. Such offsets can also be separated from gradient data in the Fourier domain.

In preferred embodiments of the third and fourth aspects of the invention, data obtained is converted to the Fourier domain. In the Fourier domain the gradient information is distinguishable from the field information because the signals due to the gradient fields are centred at twice the rotation frequency while those due to the fields are centred at the rotation frequency.

Further, information about the g_{xy} component of the gradient tensor may be distinguished from that due the diagonal gradient components, even at the same frequency, because one involves taking the Fourier transform of a cosine waveform, which is real and even, while the other involves the Fourier transform of a sine wave, which is odd and imaginary. The Fourier transformation may be implemented by means of a conventional fast Fourier transform algorithm or even simple discrete implementation via a coarse sampling of say 8 data points per cycle of the rotation.

To first-order any mismatches in the sensitivities of the field sensors or their relative alignment gives rise to contributions only at the fundamental frequency (discussed in more detail in the following) and therefore such mismatches do not determine the ultimate CMRR of the gradiometer. The final CMRR performance is determined by the resolution and precision of the data handling and the ultimate noise floor of the detectors. Accordingly, preferred

embodiments of the present invention may provide for a CMRR improved by several orders of magnitude over that of state of the art instruments.

Further, since the external gradient and field are modulated by the rotation, but DC offsets, drift or very low frequency noise in the detectors are
5 not, the rotation provided by the present invention facilitates separation of the these effects in the Fourier domain so that the DC and low frequency performance of the instrument is not limited by the performance of the detectors in these respects.

Furthermore, in the case of magnetic gradiometry, if the gradiometer is to
10 be employed in a moving platform such as an aircraft, for example, a suspension system is generally used to isolate the instrument from vibrations which would otherwise induce significant noise by moving the instrument in the earth's magnetic field. However, in previous systems, such suspension systems have proven to be extremely difficult to engineer as significant
15 isolation is required at sub 1 Hz frequencies. However, the rotating gradiometer provided by present invention modulates low frequency signals with the rotation frequency, thus facilitating the use of suspension systems which then only need to provide effective vibration isolation near the fundamental and first overtone of the rotation frequency. Such suspension
20 systems are typically significantly easier to engineer than sub-1Hz suspension systems.

In preferred embodiments of the third and fourth aspects of the invention, the at least three gradiometers are rotated at differing frequencies. Such embodiments of the invention facilitate separation of the frequency of operation
25 of each gradiometer, and further facilitate a reduction in "cross-talk" between the respective gradiometers, of the type set out in the preceding discussion. As such cross-talk effects will be occurring at differing frequencies, the effect of one gradiometer operation on another can be filtered out or removed in the frequency domain.

30 Accordingly, preferred embodiments of the present invention may yield significant advantages over previous field detection devices and methods in a number of ways. First, rotation of the gradiometer breaks the nexus between engineering tolerances (in positioning two sensors in precisely parallel planes) and the ultimate CMRR performance. Further, providing three rotating
35 gradiometers in accordance with embodiments of the third and fourth aspects of the present invention yields measurements of all five components of the

gradient tensor and all three the components of the field, with redundancy in the data, all by using only 3 field detectors. Additionally, the rotation of the gradiometer breaks the dependence of the DC performance of the instrument on the DC performance of the detectors. Embodiments of the present invention

5 also facilitate absolute value measurements of both the gradient and field components using SQUIDs. Embodiments of the present invention may also provide real time information of the condition of the SQUID in operation, for example as to whether or not the SQUID has remained in lock. As noted in the preceding discussions, previous SQUID devices have only been capable of

10 making relative field measurements, not absolute field measurements. Furthermore, embodiments of the present invention may improve the sensitivity of field detection by shifting the working point of the detectors to a higher frequency, where the noise performance of the detectors and the vibration isolation afforded by suspension systems is superior. Yet further, certain

15 embodiments of the present invention can serve to reduce the effects of cross-talk by use of distinguishable rotation frequencies.

Brief Description of the Drawings

     By way of example only, preferred embodiments of the invention will be

20 described with reference to the accompanying drawings, in which:

     Figure 1a illustrates a low temperature superconducting gradiometer in accordance with the present invention;

     Figure 1b illustrates a high temperature superconducting gradiometer in accordance with the present invention; and

25   Figure 2 illustrates a three-gradiometer magnetic field detection device in accordance with the present invention.

Description of the Invention

     Figure 1a illustrates a low temperature superconducting gradiometer 10

30 in accordance with the present invention. Gradiometer 10 comprises a first field vector sensor in the form of superconducting wire loop 11. Gradiometer 10 further comprises a second field vector sensor in the form of superconducting wire loop 12. Sensors 11 and 12 are positioned substantially coaxially so as to sense field vectors along axis 13. . Further, sensors 11 and 12 are connected

35 in a differencing arrangement, whereby a signal produced at terminals 14 stems from the difference between the magnetic field vectors sensed by

sensors 11 and 12. In accordance with the present invention, gradiometer 10 is configured to rotate about axis 15, which is at a normal to the page as viewed in Figure 1a. Rotation of the gradiometer is indicated by direction arrows 16.

Figure 1b illustrates a high temperature superconducting gradiometer 20 in accordance with the present invention. Gradiometer 20 comprises a first field vector sensor in the form of a SQUID 21. Gradiometer 10 further comprises a second field vector sensor in the form of a SQUID 22. SQUIDs 21 and 22 are mounted on a suitable magnetically transparent mount. Sensors 21 and 22 are positioned substantially coaxially so as to sense field vectors along axis 23. Further, sensors 21 and 22 are connected in a differencing arrangement (not shown), whereby a signal thus produced stems from the difference between the magnetic field vectors sensed by sensors 21 and 22. In accordance with the present invention, gradiometer 20 is configured to rotate about axis 25.

Figure 2 illustrates a three-gradiometer magnetic field detection device 30 in accordance with the present invention. The gradiometers are represented by cylinders 31, 32, 33. Cylinders 31, 32, 33 are respectively positioned on axes 41, 42, 43, and each of axes 41, 42 and 43 is not parallel with either of the other two axes. Each of gradiometers 31, 32, 33 is rotatable about unique non-parallel axes in accordance with the present invention. Operation of such a device is set out in more detail in the following.

This calculation models the sensitivity of the gradiometer instrument to the magnetic field and gradient tensor in the presence of 6 pairs of misalignment angles introduced by tilting the SQUIDs through arbitrary angles. The effects of precession are not considered explicitly but can probably be modelled to sufficient accuracy within the theory. It is demonstrated that the misalignment errors lead to contamination of the gradient and B-field components by terms that depend on the misalignments, but the effects of misalignment can be removed by the theory by introducing appropriate calibrations constants. A final solution is derived in terms of instrument coordinates and the transformations to geographic coordinates is also described.

The resulting instrument provides a measurement of the true-value of the gradient and field components which is relatively robust with respect to changes in the calibration constants.

I. OVERVIEW

The present embodiment comprises of a set of three rotating axial gradiometers or *drum gra-diometers*. In the 2-SQUID implementation the output voltages from a pair of SQUIDs operated in flux-locked loops are differenced to provide information about the magnetic field density and the first-order magnetic gradient tensor. If the SQUIDs are perfectly aligned with identical sensitivities then the difference voltage provides no information about the magnetic field density but this information is restored if the sensitivities of the SQUIDs are mismatched by a known amount. The difference voltage also provides a measurement of the first-order axial gradient provided that the contributions from higher-order gradients are sufficiently small to be ignored. This is easily seen by expanding the B-field as a Taylor series about the origin which is take to be centre point of the axial gradiometer.

A. Taylor Series for functions of three variables.

The Taylor series for a scalar function $B(x, y, z)$ about the origin is

$$B(x, y, z) = \sum_{n=0}^{\infty} \frac{1}{n!} \left(x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z} \right)^n B(0). \tag{1}$$

In the case of a 3-dimensional vector field the previous equation applies to each of the vector components, so the α-component of B at the point r, $B_\alpha(r)$, and at the origin, $B_\alpha(0)$, are related by

$$B_\alpha(r) = B_\alpha(0) + \left(x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z} \right) B_\alpha + \frac{1}{2!} \left(x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z} \right) B_\alpha$$

$$+ \frac{1}{3!} \left(x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z} \right)^3 B_\alpha + \frac{1}{4!} \left(x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z} \right)^4 B_\alpha + \cdots \tag{2}$$

If the second- and higher-order gradients of B are small then 2 may be approximated by

$$= B_\alpha(0) + \left(x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z} \right) B_\alpha \tag{3}$$

provided that r is small.

Let

$$g_{\alpha\beta} \triangleq \frac{\partial}{\partial_\beta} B_\alpha$$

denote components of the the first-order gradients of **B**. Then eq. ?? becomes simply

$$B_\alpha(\mathbf{r}) \doteq B_\alpha(0) + x g_{\alpha x} + y g_{\alpha y} + z g_{\alpha z} \qquad (4)$$

or in coordinate independent form

$$\mathbf{B}(\mathbf{r}) \doteq \mathbf{B}(0) + \hat{g}\mathbf{r} \qquad (5)$$

where $\hat{g}$ is the gradient matrix and **r** is the displacement vector to the point at which the **B** field is measured.

Using 5 if $\mathbf{r}_1$ and **r** are two points the difference in the **B**-field at these points is

$$\mathbf{B}(\mathbf{r}_1) - \mathbf{B}(\mathbf{r}) \doteq \mathbf{B}(0) + \hat{g}\mathbf{r}_1 - (\mathbf{B}(0) + \hat{g}\mathbf{r}) = \hat{g}(\mathbf{r}_1' - \mathbf{r}')$$

which is proportional to the product of the gradient matrix and the displacement between the points.

II. INSTRUMENT AND DRUM COORDINATES.

The drum gradiometers rotate about axes with orientations denoted by the vectors **U**, **V** and **W** in an *umbrella configuration* to be described in detail below. The axial gradiometers within the drums are located at the end points of **U**, **V** and **W**, and facilitate absolute value measurements of linear combinations of some of the components of the magnetic gradient tensor and magnetic field with respect to a *drum coordinate system*, C', which is fixed to the axis of rotation of each of the three drums. The systems of equations in drum coordinates are transformed to a common *instrument coordinate system*, C, and solved to yield absolute value measurements of the magnetic gradient tensor and magnetic field in instrument coordinates.

For use in geophysical exploration the final measurements have to be expressed in terms of the *geographic coordinate system*, C_G. This is easily done if the orientation of C with respect to C_G is known but in general this will require additional information from position and orientation sensors. It is possible that, in some applications, the real-time measurements of the magnetic field provided by the instrument itself, might be used to provide information about the orientation of the instrument with respect to the magnetic field of the earth. Transformations of the final measurements between the instrument coordinates and geographic coordinates are described below in section VI.

A. Instrument coordinate system, C.

The instrument coordinates system, C, is a right-handed Cartesian coordinate system defined as follows. The x-y plane is defined by the base of the instrument. The z-direction is normal to the x-y plane and forms an axis of symmetry for the instrument in the sense that the three drum gradiometers are distributed symmetrically about the z-axis. Unit vectors in the directions of the x-, y- and z-axes will be denoted i, j and k, respectively.

As mentioned above the axial gradiometers are located at the end points of **U**, **V** and **W** and rotate about the axes defined by these vectors. The direction of **U** is the vector obtained by revolving a unit vector i with its base at the origin of C about the $-y$ axis through an angle λ. The directions of **V** and **W** are obtained by revolving **U** about the z-axis through angles of $120°$ and $240°$ respectively. **U**, **V** and **W** then form a type of *umbrella configuration* being disposed symmetrically about the z-axis and tilted above the x-y plane through an *umbrella angle* angle λ. Hence

$$\mathbf{U} = (L_u \cos \lambda, 0, L_u \sin \lambda) \tag{6}$$

$$\mathbf{V} = \left(-\frac{1}{2} L_v \cos \lambda, \frac{\sqrt{3}}{2} L_v \cos \lambda, L_v \sin \lambda \right) \tag{7}$$

$$\mathbf{W} = \left(-\frac{1}{2} L_w \cos \lambda, -\frac{\sqrt{3}}{2} L_w \cos \lambda, L_w \sin \lambda \right) \tag{8}$$

in which L_u, L_v and L_w denote the distances of the axial gradiometers from the origin of C.

1. Properties of **U**, **V** *and* **W**.

● Note that these unit vectors form an orthogonal system only if

$$\mathbf{U} \cdot \mathbf{V} = \mathbf{V} \cdot \mathbf{W} = \mathbf{W} \cdot \mathbf{U} = 0$$

i.e. if

$$\frac{1}{4} - \frac{3}{4} \cos 2\lambda = 0$$

$$\Rightarrow \lambda = \frac{1}{2} \arccos \frac{1}{3} \doteq 0.615\,48 \,\mathrm{rad} \doteq 35.\,3° \tag{9}$$

- In the   instrument $\lambda = 45°$ so from equations 6, 7 and 8

$$\mathbf{U} = \left(\cos\frac{\pi}{4}, 0, \sin\frac{\pi}{4}\right) = L_u \left(\begin{array}{ccc} \frac{1}{2}\sqrt{2} & 0 & \frac{1}{2}\sqrt{2} \end{array}\right) \tag{10}$$

$$\mathbf{V} = \left(-\frac{1}{2}\cos\frac{\pi}{4}, \frac{\sqrt{3}}{2}\cos\frac{\pi}{4}, \sin\frac{\pi}{4}\right) = L_v \left(\begin{array}{ccc} -\frac{1}{4}\sqrt{2} & \frac{1}{4}\sqrt{6} & \frac{1}{2}\sqrt{2} \end{array}\right) \tag{11}$$

$$\mathbf{W} = \left(-\frac{1}{2}\cos\frac{\pi}{4}, -\frac{\sqrt{3}}{2}\cos\frac{\pi}{4}, \sin\frac{\pi}{4}\right) = L_w \left(\begin{array}{ccc} -\frac{1}{4}\sqrt{2} & -\frac{1}{4}\sqrt{6} & \frac{1}{2}\sqrt{2} \end{array}\right) \tag{12}$$

2.   Unit drum vectors

It will prove useful to define the *unit drum vectors,* u_0, v_0 and w_0, that are normal to $\mathbf{U}$, $\mathbf{V}$ and $\mathbf{W}$ respectively. Vector u_0 is the unit vector obtained by revolving a unit vector i with its base at the origin of C about the $+y$ axis through an angle $(90 - \lambda)$ degrees. Vectors v_0 and w_0 are obtained by revolving u_0 about the z-axis through angles of $120°$ and $240°$ respectively. In instrument coordinates

$$\mathbf{u_0} = \left(\begin{array}{ccc} \sin\lambda & 0 & -\cos\lambda \end{array}\right) \tag{13}$$

$$\mathbf{v_0} = \left(\begin{array}{ccc} -\frac{1}{2}\sin\lambda & \frac{\sqrt{3}}{2}(\sin\lambda) & -\cos\lambda \end{array}\right) \tag{14}$$

$$\mathbf{w_0} = \left(\begin{array}{ccc} -\frac{1}{2}\sin\lambda & -\frac{\sqrt{3}}{2}(\sin\lambda) & -\cos\lambda \end{array}\right) \tag{15}$$

N.B.

$$\mathbf{u_0} \cdot \mathbf{U} = \left(\begin{array}{ccc} \sin\lambda & 0 & -\cos\lambda \end{array}\right) \cdot L_u (\cos\lambda, 0, \sin\lambda) = 0$$

$$\mathbf{v_0} \cdot \mathbf{V} = \left(\begin{array}{ccc} -\frac{1}{2}\sin\lambda & \frac{\sqrt{3}}{2}(\sin\lambda) & -\cos\lambda \end{array}\right) \cdot L_v \left(-\frac{1}{2}\cos\lambda, \frac{\sqrt{3}}{2}\cos\lambda, \sin\lambda\right) = 0$$

$$\mathbf{w_0} \cdot \mathbf{W} = \left(\begin{array}{ccc} -\frac{1}{2}\sin\lambda & -\frac{\sqrt{3}}{2}(\sin\lambda) & -\cos\lambda \end{array}\right) \cdot L_w \left(-\frac{1}{2}\cos\lambda, -\frac{\sqrt{3}}{2}\cos\lambda, \sin\lambda\right) = 0$$

as required.

B.   Drum coordinate systems C'

As mentioned earlier we introduce 3 drum coordinate systems, C', C'' and C''' associated with the each of the three drums rotating about $\mathbf{U}$, $\mathbf{V}$ and $\mathbf{W}$ respectively. Let x', y' and z' denote unit vectors in the directions of the x', y' and z' axes of the drum coordinate systems. C' is defined so that $x' = u_0$, $y' = j$, $z' = \mathbf{U}/L_u$ and the origins of C'_U and C coincide. With these definitions

C'_U is a right-handed coordinate system which would become equivalent to C if the umbrella angle was set to 90°.

C'' and C''' are then defined by rotating C' about the z-axis through angles 120° and 240° so that in C'', $\mathbf{x}'' = \mathbf{v}_0$ and $\mathbf{z}'' = \mathbf{V}/L_v$ while in C''', $\mathbf{x}''' = \mathbf{w}_0$ and $\mathbf{z}''' = \mathbf{W}/L_w$.

1. Unit gradiometer vectors.

The instantaneous direction of the rotating axial gradiometers will be defined by *unit gradiometer vectors* $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$, which remain normal to $\mathbf{U}$, $\mathbf{V}$ and $\mathbf{W}$ and undergo continuous rotation about the end points of $\mathbf{U}$, $\mathbf{V}$ and $\mathbf{W}$ with angular frequencies ω_u, ω_v and ω_w respectively. θ_u, θ_v and θ_w will denote respectively the angular displacements of the vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ at time t, from the positions of $\mathbf{u}_0$, $\mathbf{v}_0$ and $\mathbf{w}_0$ defined above, so that $\theta_u = \omega_u t$, $\theta_v = \omega_v t$ and $\theta_w = \omega_w t$. In terms of the drum coordinates C'_U, C'_V and C'_W the unit gradiometer vectors $\mathbf{u}'$, $\mathbf{v}'$ and $\mathbf{w}'$ assume the same form

$$\mathbf{u}' = (\cos\theta_u, \sin\theta_u, 0) \tag{16}$$

$$\mathbf{v}' = (\cos\theta_v, \sin\theta_v, 0) \tag{17}$$

$$\mathbf{w}' = (\cos\theta_w, \sin\theta_w, 0) \tag{18}$$

C. Coordinate transformations between C and the C' systems.

In general if $\mathbf{P}$ and $\mathbf{P}'$ denote the components of a vector in the frames C and C' respectively and if C is mapped to C' by a positive (right-handed) rotation through an angle κ about an axis k ($k = x, y, z$) then

$$\mathbf{P}' = \widehat{R}_k(\kappa)\,\mathbf{P} \tag{19}$$

where $\widehat{R}_k(\kappa)$ denotes the standard rotation matrices

$$\widehat{R}_x(\kappa) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\kappa & \sin\kappa \\ 0 & -\sin\kappa & \cos\kappa \end{pmatrix} \tag{20}$$

$$\widehat{R}_y(\kappa) = \begin{pmatrix} \cos\kappa & 0 & -\sin\kappa \\ 0 & 1 & 0 \\ \sin\kappa & 0 & \cos\kappa \end{pmatrix} \tag{21}$$

$$\widehat{R}_z(\kappa) = \begin{pmatrix} \cos\kappa & \sin\kappa & 0 \\ -\sin\kappa & \cos\kappa & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{22}$$

Also if $\widehat{g}$ and $\widehat{g}'$ are the components of a Cartesian tensor in the frames C and C' respectively then

$$\widehat{g}' = \widehat{R}_k(\kappa)\,\widehat{g}\,\widehat{R^T}_k(\kappa). \tag{23}$$

For example a rotation about the z-axis through angle κ is

$$\begin{pmatrix} g'_{11} & g'_1 & g'_{13} \\ g'_1 & g' & g'_3 \\ g'_{13} & g'_3 & g'_{33} \end{pmatrix}$$

$$= \begin{pmatrix} \cos\kappa & \sin\kappa & 0 \\ -\sin\kappa & \cos\kappa & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} g_{11} & g_1 & g_{13} \\ g_1 & g & g_3 \\ g_{13} & g_3 & g_{33} \end{pmatrix} \begin{pmatrix} \cos\kappa & -\sin\kappa & 0 \\ \sin\kappa & \cos\kappa & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and hence

$$\left.\begin{aligned} g'_{11} &= \tfrac{1}{2}(g_{11}+g) + \tfrac{1}{2}(g_{11}-g)\cos 2\kappa + g_1\sin 2\kappa \\ g' &= \tfrac{1}{2}(g_{11}+g) - \tfrac{1}{2}(g_{11}-g)(\cos 2\kappa) - g_1\sin 2\kappa \\ g'_{33} &= g_{33} \\ g'_1 &= -\tfrac{1}{2}(g_{11}-g)(\sin 2\kappa) + g_1\cos 2\kappa \\ g'_3 &= g_3\cos\kappa - g_{13}\sin\kappa \\ g'_{13} &= g_{13}\cos\kappa + g_3\sin\kappa \end{aligned}\right\} \tag{24}$$

1. Transformations $C \rightleftarrows C'$.

When $x' = u_0, y' = j$ and $z' = U/L_u$, C is mapped onto C' by a rotation about y through an angle $(\pi/2 - \lambda)$. Hence if $\mathbf{B}$ and $\mathbf{B}'$ denote the components of a vector in the frames C and C'

respectively then

$$\mathbf{B}' = \begin{pmatrix} \sin\lambda & 0 & -\cos\lambda \\ 0 & 1 & 0 \\ \cos\lambda & 0 & \sin\lambda \end{pmatrix} \mathbf{B} \tag{25}$$

$$\mathbf{B} = \begin{pmatrix} \sin\lambda & 0 & \cos\lambda \\ 0 & 1 & 0 \\ -\cos\lambda & 0 & \sin\lambda \end{pmatrix} \mathbf{B}' \tag{26}$$

N.B. If $\lambda = \pi/4$ then

$$\mathbf{B}' = \begin{pmatrix} \frac{1}{2}\sqrt{2} & 0 & -\frac{1}{2}\sqrt{2} \\ 0 & 1 & 0 \\ \frac{1}{2}\sqrt{2} & 0 & \frac{1}{2}\sqrt{2} \end{pmatrix} \mathbf{B} \tag{27}$$

while if $\lambda = \pi/2$ 25 and 26 give that $\mathbf{P}' = \mathbf{P}$, i.e. the coordinate systems C and C_U' coincide, as required. Also, using equation 6,

$$\mathbf{U}' = \begin{pmatrix} \sin\lambda & 0 & -\cos\lambda \\ 0 & 1 & 0 \\ \cos\lambda & 0 & \sin\lambda \end{pmatrix} \mathbf{U}$$

$$= \begin{pmatrix} \sin\lambda & 0 & -\cos\lambda \\ 0 & 1 & 0 \\ \cos\lambda & 0 & \sin\lambda \end{pmatrix} \begin{pmatrix} L_u\cos\lambda \\ 0 \\ L_u\sin\lambda \end{pmatrix}$$

$$= L_u \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

as required.

Using 26 the unit gradiometer vector

$$\mathbf{u}' = \begin{pmatrix} \cos\theta_u \\ \sin\theta_u \\ 0 \end{pmatrix}$$

becomes

$$\mathbf{u} = \begin{pmatrix} \sin\lambda & 0 & \cos\lambda \\ 0 & 1 & 0 \\ -\cos\lambda & 0 & \sin\lambda \end{pmatrix} \begin{pmatrix} \cos\theta_u \\ \sin\theta_u \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\theta_u\sin\lambda \\ \sin\theta_u \\ -\cos\theta_u\cos\lambda \end{pmatrix} \tag{28}$$

in C. Observe

$$\mathbf{u} \cdot \mathbf{U} = \begin{pmatrix} \cos\theta_u \sin\lambda \\ \sin\theta_u \\ -\cos\theta_u \cos\lambda \end{pmatrix} \cdot \begin{pmatrix} L_u \cos\lambda \\ 0 \\ L_u \sin\lambda \end{pmatrix} = 0$$

so $\mathbf{u} \perp \mathbf{U}$ as required.

a. *Gradient Transformation.* The transformations of the gradient tensor from C to C' follows from the use of the transformation matrix in 25, with the general transformation equation, 23, i.e.

$$\begin{pmatrix} g'_{11} & g'_1 & g'_{13} \\ g'_1 & g' & g'_3 \\ g'_{13} & g'_3 & g'_{33} \end{pmatrix} = \begin{pmatrix} \sin\lambda & 0 & -\cos\lambda \\ 0 & 1 & 0 \\ \cos\lambda & 0 & \sin\lambda \end{pmatrix} \begin{pmatrix} g_{11} & g_1 & g_{13} \\ g_1 & g & g_3 \\ g_{13} & g_3 & g_{33} \end{pmatrix} \begin{pmatrix} \sin\lambda & 0 & \cos\lambda \\ 0 & 1 & 0 \\ -\cos\lambda & 0 & \sin\lambda \end{pmatrix} \quad (29)$$

$$\Rightarrow \left. \begin{aligned} g'_{11} &= \tfrac{1}{}\left(g_{11} + g_{33} - g_{11}\cos 2\lambda + g_{33}\cos 2\lambda - 2g_{13}\sin 2\lambda\right) \\ g' &= g \\ g'_{33} &= \tfrac{1}{}\left(g_{11} + g_{33} + g_{11}\cos 2\lambda - g_{33}\cos 2\lambda + 2g_{13}\sin 2\lambda\right) \\ g'_1 &= \tfrac{1}{}\sqrt{2}\left(g_1 - g_3\right) \\ g'_{13} &= -\tfrac{1}{}\left(2g_{13}\cos 2\lambda - g_{11}\sin 2\lambda + g_{33}\sin 2\lambda\right) \\ g'_3 &= \tfrac{1}{}\sqrt{2}\left(g_1 + g_3\right) \end{aligned} \right\} \quad (30)$$

In the special case $\lambda = \pi/4$

$$\left. \begin{aligned} g'_{11} &= \tfrac{1}{}g_{11} - g_{13} + \tfrac{1}{}g_{33} \\ g' &= g \\ g'_{33} &= \tfrac{1}{}g_{11} + g_{13} + \tfrac{1}{}g_{33} \\ g'_1 &= \tfrac{1}{}\sqrt{2}\left(g_1 - g_3\right) \\ g'_{13} &= \tfrac{1}{}g_{11} - \tfrac{1}{}g_{33} \\ g'_3 &= \tfrac{1}{}\sqrt{2}\left(g_1 + g_3\right) \end{aligned} \right\} \quad (31)$$

2. *Transformations $C \rightleftarrows C''$.*

When $\mathbf{x}'' = \mathbf{v}_0$ and $\mathbf{z}'' = \mathbf{V}/L_{\mathbf{v}}$, C is mapped onto C'' by a rotation about z through an angle $2\pi/3$ followed by a rotation about y through an angle $(\pi/2 - \lambda)$.

$$\mathbf{B}'' = \begin{pmatrix} -\frac{1}{2}\sin\lambda & \frac{1}{2}\sqrt{3}\,(\sin\lambda) & -\cos\lambda \\ -\frac{1}{2}\sqrt{3} & -\frac{1}{2} & 0 \\ -\frac{1}{2}\cos\lambda & \frac{1}{2}\sqrt{3}\,(\cos\lambda) & \sin\lambda \end{pmatrix}\mathbf{B} \qquad (32)$$

$$\mathbf{B} = \begin{pmatrix} -\frac{1}{2}\sin\lambda & -\frac{1}{2}\sqrt{3} & -\frac{1}{2}\cos\lambda \\ \frac{1}{2}\sqrt{3}\,(\sin\lambda) & -\frac{1}{2} & \frac{1}{2}\sqrt{3}\,(\cos\lambda) \\ -\cos\lambda & 0 & \sin\lambda \end{pmatrix}\mathbf{B}'' \qquad (33)$$

If $\lambda = \pi/4$

$$\mathbf{B}'' = \begin{pmatrix} -\frac{1}{4}\sqrt{2} & \frac{1}{4}\sqrt{6} & -\frac{1}{2}\sqrt{2} \\ -\frac{1}{2}\sqrt{3} & -\frac{1}{2} & 0 \\ -\frac{1}{4}\sqrt{2} & \frac{1}{4}\sqrt{6} & \frac{1}{2}\sqrt{2} \end{pmatrix}\mathbf{B}.$$

Using 33 the unit gradiometer vector

$$\mathbf{v}'' = \begin{pmatrix} \cos\theta_v \\ \sin\theta_v \\ 0 \end{pmatrix}$$

becomes

$$\mathbf{v} = \begin{pmatrix} -\frac{1}{2}\sin\lambda & -\frac{1}{2}\sqrt{3} & -\frac{1}{2}\cos\lambda \\ \frac{1}{2}\sqrt{3}\,(\sin\lambda) & -\frac{1}{2} & \frac{1}{2}\sqrt{3}\,(\cos\lambda) \\ -\cos\lambda & 0 & \sin\lambda \end{pmatrix}\begin{pmatrix} \cos\theta_v \\ \sin\theta_v \\ 0 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}\sin\lambda\cos\theta_v - \frac{1}{2}\sqrt{3}\,(\sin\theta_v) \\ -\frac{1}{2}\sin\theta_v + \frac{1}{2}\sqrt{3}\,(\sin\lambda\cos\theta_v) \\ -\cos\lambda\cos\theta_v \end{pmatrix} \qquad (34)$$

in C.

Observe that

$$\mathbf{v} \cdot \mathbf{V} = 0$$

confirming that $v \perp V$ as required. Also using 6

$$
\mathbf{V}'' = \begin{pmatrix} -\frac{1}{2}\sin\lambda & \frac{1}{2}\sqrt{3}\,(\sin\lambda) & -\cos\lambda \\ -\frac{1}{2}\sqrt{3} & -\frac{1}{2} & 0 \\ -\frac{1}{2}\cos\lambda & \frac{1}{2}\sqrt{3}\,(\cos\lambda) & \sin\lambda \end{pmatrix} \mathbf{V}
$$

$$
= \begin{pmatrix} -\frac{1}{2}\sin\lambda & \frac{1}{2}\sqrt{3}\,(\sin\lambda) & -\cos\lambda \\ -\frac{1}{2}\sqrt{3} & -\frac{1}{2} & 0 \\ -\frac{1}{2}\cos\lambda & \frac{1}{2}\sqrt{3}\,(\cos\lambda) & \sin\lambda \end{pmatrix} \begin{pmatrix} -\frac{1}{2}L_v\cos\lambda \\ \frac{\sqrt{3}}{2}L_v\cos\lambda \\ L_v\sin\lambda \end{pmatrix}
$$

$$
= L_v \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}
$$

as required.

   $a.$ *Gradient Transformation.*   The transformations of the gradient tensor from C to C' follows from the use of the transformation matrix in 32, with the general transformation equation, 23, i.e.

$$
\begin{pmatrix} g''_{11} & g''_1 & g''_{13} \\ g''_1 & g'' & g''_3 \\ g''_{13} & g''_3 & g''_{33} \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}\sin\lambda & \frac{\sqrt{3}}{2}\,(\sin\lambda) & -\cos\lambda \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{2}\cos\lambda & \frac{\sqrt{3}}{2}\,(\cos\lambda) & \sin\lambda \end{pmatrix} \begin{pmatrix} g_{11} & g_1 & g_{13} \\ g_1 & g & g_3 \\ g_{13} & g_3 & g_{33} \end{pmatrix} \begin{pmatrix} -\frac{1}{2}\sin\lambda & -\frac{\sqrt{3}}{2} & -\frac{1}{2}\cos\lambda \\ \frac{\sqrt{3}}{2}\,(\sin\lambda) & -\frac{1}{2} & \frac{\sqrt{3}}{2}\,(\cos\lambda) \\ -\cos\lambda & 0 & \sin\lambda \end{pmatrix}
$$

$$(35)$$

$$g''_{11} = \tfrac{1}{8}g_{11} + \tfrac{3}{8}g \quad + \tfrac{1}{}g_{33} - \tfrac{1}{8}g_{11}\cos 2\lambda - \tfrac{3}{8}g \quad \cos 2\lambda + \tfrac{1}{}g_{33}\cos 2\lambda$$
$$+\tfrac{1}{}g_{13}\sin 2\lambda - \tfrac{1}{4}g_1 \ \sqrt{3} + \tfrac{1}{4}g_1 \ \sqrt{3}\cos 2\lambda - \tfrac{1}{}g \ _3\sqrt{3}\sin 2\lambda$$

$$g'' \ = \tfrac{1}{4}g \ + \tfrac{1}{4}g_1 \ \sqrt{3} + \sqrt{3}\left(\tfrac{1}{4}g_1 \ + \tfrac{1}{4}g_{11}\sqrt{3}\right)$$

$$g''_{33} = \tfrac{1}{8}g_{11} + \tfrac{3}{8}g \quad + \tfrac{1}{}g_{33} + \tfrac{1}{8}g_{11}\cos 2\lambda + \tfrac{3}{8}g \quad \cos 2\lambda - \tfrac{1}{}g_{33}\cos 2\lambda$$
$$-\tfrac{1}{}g_{13}\sin 2\lambda - \tfrac{1}{4}g_1 \ \sqrt{3} - \tfrac{1}{4}g_1 \ \sqrt{3}\cos 2\lambda + \tfrac{1}{}g \ _3\sqrt{3}\sin 2\lambda$$

$$\Rightarrow \quad g''_1 \ = \tfrac{1}{}(\cos\lambda)\,g \ _3 - \tfrac{1}{}(\sin\lambda)\,g_1 \ + \tfrac{1}{}(\cos\lambda)\,g_{13}\sqrt{3}$$
$$+\tfrac{1}{4}(\sin\lambda)\,g_{11}\sqrt{3} - \tfrac{1}{4}(\sin\lambda)\,g \ \sqrt{3}$$

$$g''_{13} = \tfrac{1}{}g_{13}\cos 2\lambda + \tfrac{1}{8}g_{11}\sin 2\lambda + \tfrac{3}{8}g \quad \sin 2\lambda - \tfrac{1}{}g_{33}\sin 2\lambda$$
$$-\tfrac{1}{}g \ _3\sqrt{3}\cos 2\lambda - \tfrac{1}{4}g_1 \ \sqrt{3}\sin 2\lambda$$

$$g''_3 = -\tfrac{1}{}(\cos\lambda)\,g_1 \ - \tfrac{1}{}(\sin\lambda)\,g \ _3 + \tfrac{1}{4}(\cos\lambda)\,g_{11}\sqrt{3}$$
$$-\tfrac{1}{4}(\cos\lambda)\,g \ \sqrt{3} - \tfrac{1}{}(\sin\lambda)\,g_{13}\sqrt{3}$$

\right\} \quad (36)

In the special case $\lambda = \pi/4$

$$g''_{11} = \tfrac{1}{8}g_{11} + \tfrac{1}{}g_{13} + \tfrac{3}{8}g \quad + \tfrac{1}{}g_{33} - \tfrac{1}{4}\sqrt{3}g_1 \ - \tfrac{1}{}g \ _3\sqrt{3}$$
$$g'' \ = \tfrac{3}{4}g_{11} + \tfrac{1}{4}g \quad + \tfrac{1}{}g_1 \ \sqrt{3}$$
$$g''_{33} = \tfrac{1}{8}g_{11} - \tfrac{1}{}g_{13} + \tfrac{3}{8}g \quad + \tfrac{1}{}g_{33} - \tfrac{1}{4}\sqrt{3}g_1 \ + \tfrac{1}{}\sqrt{3}g \ _3$$
$$g''_1 \ = -\tfrac{1}{4}\sqrt{2}g_1 \ + \tfrac{1}{4}\sqrt{2}g \ _3 + \tfrac{1}{8}\sqrt{6}g_{11} + \tfrac{1}{4}\sqrt{6}g_{13} - \tfrac{1}{8}\sqrt{6}g$$
$$g''_{13} = \tfrac{1}{8}g_{11} + \tfrac{3}{8}g \quad - \tfrac{1}{}g_{33} - \tfrac{1}{4}\sqrt{3}g_1$$
$$g''_3 = -\tfrac{1}{4}\sqrt{2}g_1 \ - \tfrac{1}{4}\sqrt{2}g \ _3 + \tfrac{1}{8}\sqrt{6}g_{11} - \tfrac{1}{4}\sqrt{6}g_{13} - \tfrac{1}{8}\sqrt{6}g$$

\right\} \quad (37)

3. Transformations $C \rightleftarrows C'''$.

When $\mathbf{x}''' = \mathbf{w}_0$ and $\mathbf{z}''' = \mathbf{W}/L_w$, C is mapped onto C''' by a rotation about z through an angle $4\pi/3$ followed by a rotation about y through an angle $(\pi/2 - \lambda)$. Hence

$$\mathbf{B}''' = \begin{pmatrix} -\frac{1}{2}\sin\lambda & -\frac{1}{2}\sqrt{3}\,(\sin\lambda) & -\cos\lambda \\ \frac{1}{2}\sqrt{3} & -\frac{1}{2} & 0 \\ -\frac{1}{2}\cos\lambda & -\frac{1}{2}\sqrt{3}\,(\cos\lambda) & \sin\lambda \end{pmatrix} \mathbf{B} \tag{38}$$

$$\mathbf{B} = \begin{pmatrix} -\frac{1}{2}\sin\lambda & \frac{1}{2}\sqrt{3} & -\frac{1}{2}\cos\lambda \\ -\frac{1}{2}\sqrt{3}\,(\sin\lambda) & -\frac{1}{2} & -\frac{1}{2}\sqrt{3}\,(\cos\lambda) \\ -\cos\lambda & 0 & \sin\lambda \end{pmatrix} \mathbf{B}''' \tag{39}$$

If $\lambda = \pi/4$

$$\mathbf{B}''' = \begin{pmatrix} -\frac{1}{4}\sqrt{2} & -\frac{1}{4}\sqrt{6} & -\frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{3} & -\frac{1}{2} & 0 \\ -\frac{1}{4}\sqrt{2} & -\frac{1}{4}\sqrt{6} & \frac{1}{2}\sqrt{2} \end{pmatrix} \mathbf{B}. \tag{40}$$

Using 39 the unit gradiometer vector

$$\mathbf{w}''' = \begin{pmatrix} \cos\theta_w \\ \sin\theta_w \\ 0 \end{pmatrix}$$

becomes

$$\mathbf{w} = \begin{pmatrix} -\frac{1}{2}\sin\lambda & \frac{1}{2}\sqrt{3} & -\frac{1}{2}\cos\lambda \\ -\frac{1}{2}\sqrt{3}\,(\sin\lambda) & -\frac{1}{2} & -\frac{1}{2}\sqrt{3}\,(\cos\lambda) \\ -\cos\lambda & 0 & \sin\lambda \end{pmatrix} \begin{pmatrix} \cos\theta_v \\ \sin\theta_v \\ 0 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}\sin\lambda\cos\theta_v + \frac{1}{2}\sqrt{3}\,(\sin\theta_v) \\ -\frac{1}{2}\sin\theta_v - \frac{1}{2}\sqrt{3}\,(\sin\lambda\cos\theta_v) \\ -\cos\lambda\cos\theta_v \end{pmatrix} \tag{41}$$

in C.

Observe that

$$\mathbf{w} \cdot \mathbf{W} = 0$$

confirming that $w \perp W$ as required. Also using equation 8

$$\mathbf{W}''' = \begin{pmatrix} -\tfrac{1}{2}\sin\lambda & -\tfrac{1}{2}\sqrt{3}(\sin\lambda) & -\cos\lambda \\ \tfrac{1}{2}\sqrt{3} & -\tfrac{1}{2} & 0 \\ -\tfrac{1}{2}\cos\lambda & -\tfrac{1}{2}\sqrt{3}(\cos\lambda) & \sin\lambda \end{pmatrix} \mathbf{W}$$

$$= \begin{pmatrix} -\tfrac{1}{2}\sin\lambda & -\tfrac{1}{2}\sqrt{3}(\sin\lambda) & -\cos\lambda \\ \tfrac{1}{2}\sqrt{3} & -\tfrac{1}{2} & 0 \\ -\tfrac{1}{2}\cos\lambda & -\tfrac{1}{2}\sqrt{3}(\cos\lambda) & \sin\lambda \end{pmatrix} \begin{pmatrix} -\tfrac{1}{2}L_w\cos\lambda \\ -\tfrac{\sqrt{3}}{2}L_w\cos\lambda \\ L_w\sin\lambda \end{pmatrix}$$

$$= L_w \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

as required.

     a. Gradient Transformation. The transformations of the gradient tensor from $\mathcal{C}$ to $\mathcal{C}'''$ follows from the use of the transformation matrix in 38, with the general transformation equation, 23, i.e.

$$\begin{pmatrix} g_{11}''' & g_{1}''' & g_{13}''' \\ g_{1}''' & g''' & g''_{3}''' \\ g_{13}''' & g''_{3} & g_{33}''' \end{pmatrix} = \begin{pmatrix} -\tfrac{1}{2}\sin\lambda & -\tfrac{\sqrt{3}}{2}(\sin\lambda) & -\cos\lambda \\ \tfrac{\sqrt{3}}{2} & -\tfrac{1}{2} & 0 \\ -\tfrac{1}{2}\cos\lambda & -\tfrac{\sqrt{3}}{2}(\cos\lambda) & \sin\lambda \end{pmatrix} \begin{pmatrix} g_{11} & g_{1} & g_{13} \\ g_{1} & g & g_{3} \\ g_{13} & g_{3} & g_{33} \end{pmatrix} \begin{pmatrix} -\tfrac{1}{2}\sin\lambda & \tfrac{1}{2}\sqrt{3} & -\tfrac{1}{2}\cos\lambda \\ -\tfrac{\sqrt{3}}{2}(\sin\lambda) & -\tfrac{1}{2} & -\tfrac{\sqrt{3}}{2}(\cos\lambda) \\ -\cos\lambda & 0 & \sin\lambda \end{pmatrix}$$

$$\text{(42)}$$

$$g_{11}''' = \tfrac{1}{8}g_{11} + \tfrac{3}{8}g + \tfrac{1}{8}g_{33} - \tfrac{1}{8}g_{11}\cos 2\lambda - \tfrac{3}{8}g\cos 2\lambda + \tfrac{1}{8}g_{33}\cos 2\lambda$$
$$+ \tfrac{1}{4}g_{13}\sin 2\lambda + \tfrac{1}{4}g_1\sqrt{3} - \tfrac{1}{4}g_1\sqrt{3}\cos 2\lambda + \tfrac{1}{4}g\,3\sqrt{3}\sin 2\lambda$$

$$g''' = \tfrac{3}{4}g_{11} + \tfrac{1}{4}g - \tfrac{1}{4}g_1\sqrt{3}$$

$$g_{33}''' = \tfrac{1}{8}g_{11} + \tfrac{3}{8}g + \tfrac{1}{8}g_{33} + \tfrac{1}{8}g_{11}\cos 2\lambda + \tfrac{3}{8}g\cos 2\lambda - \tfrac{1}{8}g_{33}\cos 2\lambda$$
$$- \tfrac{1}{4}g_{13}\sin 2\lambda + \tfrac{1}{4}g_1\sqrt{3} + \tfrac{1}{4}g_1\sqrt{3}\cos 2\lambda - \tfrac{1}{4}g\,3\sqrt{3}\sin 2\lambda$$

$$\Rightarrow \quad g_1''' = \tfrac{1}{2}(\cos\lambda)\,g\,3 - \tfrac{1}{2}(\sin\lambda)\,g_1 - \tfrac{1}{4}(\cos\lambda)\,g_{13}\sqrt{3}$$
$$- \tfrac{1}{4}(\sin\lambda)\,g_{11}\sqrt{3} + \tfrac{1}{4}(\sin\lambda)\,g\sqrt{3}$$

$$g_{13}''' = \tfrac{1}{2}g_{13}\cos 2\lambda + \tfrac{1}{8}g_{11}\sin 2\lambda + \tfrac{3}{8}g\sin 2\lambda$$
$$- \tfrac{1}{2}g_{33}\sin 2\lambda + \tfrac{1}{4}g\,3\sqrt{3}\cos 2\lambda + \tfrac{1}{4}g_1\sqrt{3}\sin 2\lambda$$

$$g'''_3 = -\tfrac{1}{2}(\cos\lambda)\,g_1 - \tfrac{1}{2}(\sin\lambda)\,g\,3 - \tfrac{1}{4}(\cos\lambda)\,g_{11}\sqrt{3}$$
$$+ \tfrac{1}{4}(\cos\lambda)\,g\sqrt{3} + \tfrac{1}{2}(\sin\lambda)\,g_{13}\sqrt{3}$$

$$\left.\right\} \quad (43)$$

In the special case $\lambda = \pi/4$

$$g_{11}''' = \tfrac{1}{8}g_{11} + \tfrac{1}{2}g_{13} + \tfrac{3}{8}g + \tfrac{1}{8}g_{33} + \tfrac{1}{4}\sqrt{3}g_1 + \tfrac{1}{4}\sqrt{3}g\,3$$
$$g''' = \tfrac{3}{4}g_{11} + \tfrac{1}{4}g - \tfrac{1}{4}\sqrt{3}g_1$$
$$g_{33}''' = \tfrac{1}{8}g_{11} - \tfrac{1}{2}g_{13} + \tfrac{3}{8}g + \tfrac{1}{8}g_{33} + \tfrac{1}{4}\sqrt{3}g_1 - \tfrac{1}{4}\sqrt{3}g\,3$$
$$g_1''' = -\tfrac{1}{4}\sqrt{2}g_1 + \tfrac{1}{4}\sqrt{2}g\,3 - \tfrac{1}{8}\sqrt{6}g_{11} - \tfrac{1}{4}\sqrt{6}g_{13} + \tfrac{1}{8}\sqrt{6}g$$
$$g_{13}''' = \tfrac{1}{8}g_{11} + \tfrac{3}{8}g - \tfrac{1}{2}g_{33} + \tfrac{1}{4}\sqrt{3}g_1$$
$$g'''_3 = -\tfrac{1}{4}\sqrt{2}g_1 - \tfrac{1}{4}\sqrt{2}g\,3 - \tfrac{1}{8}\sqrt{6}g_{11} + \tfrac{1}{4}\sqrt{6}g_{13} + \tfrac{1}{8}\sqrt{6}g$$

$$\left.\right\} \quad (44)$$

D. SQUID Configuration

The SQUIDs composing the axial gradiometer rotating about $\mathbf{U}$ are located at the endpoints of the vectors $\mathbf{r} = \mathbf{U} \pm \rho\mathbf{u}$, where $\rho = d/2$ and d is the total distance between the SQUIDs. In

drum coordinates

$$\mathbf{r}'_+ = \mathbf{U}' + \rho\mathbf{u}' = \begin{pmatrix} 0 \\ 0 \\ L_u \end{pmatrix} + \begin{pmatrix} \rho\cos\theta_u \\ \rho\sin\theta_u \\ 0 \end{pmatrix} = \begin{pmatrix} \rho\cos\theta_u \\ \rho\sin\theta_u \\ L_u \end{pmatrix} \tag{45}$$

$$\mathbf{r}'_- = \mathbf{U}' - \rho\mathbf{u}' = \begin{pmatrix} 0 \\ 0 \\ L_u \end{pmatrix} - \begin{pmatrix} \rho\cos\theta_u \\ \rho\sin\theta_u \\ 0 \end{pmatrix} = \begin{pmatrix} -\rho\cos\theta_u \\ -\rho\sin\theta_u \\ L_u \end{pmatrix} \tag{46}$$

Using eqs. 45 and 46 in equation 5 the magnetic field at the SQUIDs may be written

$$
\begin{aligned}
\mathbf{B}'_+ (\mathbf{r}'_+) &= \begin{pmatrix} B'_x \\ B'_y \\ B'_z \end{pmatrix} + \begin{pmatrix} g'_{xx} & g'_{xy} & g'_{xz} \\ g'_{yx} & g'_{yy} & g'_{yz} \\ g'_{zx} & g'_{yz} & g'_{zz} \end{pmatrix} \begin{pmatrix} \rho\cos\theta_u \\ \rho\sin\theta_u \\ L_u \end{pmatrix} \\
&= \begin{pmatrix} B'_x + L_u g'_{xz} + \rho\left(\sin\theta_u\right) g'_{xy} + \rho\left(\cos\theta_u\right) g'_{xx} \\ B'_y + L_u g'_{yz} + \rho\left(\cos\theta_u\right) g'_{xy} + \rho\left(\sin\theta_u\right) g'_{yy} \\ B'_z + \rho\left(\cos\theta_u\right) g'_{xz} + \rho\left(\sin\theta_u\right) g'_{yz} + L_u g'_{zz} \end{pmatrix}
\end{aligned} \tag{47}
$$

$$
\begin{aligned}
\mathbf{B}'_- (\mathbf{r}'_-) &= \begin{pmatrix} B'_x \\ B'_y \\ B'_z \end{pmatrix} + \begin{pmatrix} g'_{xx} & g'_{xy} & g'_{xz} \\ g'_{yx} & g'_{yy} & g'_{yz} \\ g'_{zx} & g'_{zy} & g'_{zz} \end{pmatrix} \begin{pmatrix} -\rho\cos\theta_u \\ -\rho\sin\theta_u \\ L_u \end{pmatrix} \\
&= \begin{pmatrix} B'_x + L_u g'_{xz} - \rho\left(\sin\theta_u\right) g'_{xy} - \rho\left(\cos\theta_u\right) g'_{xx} \\ B'_y + L_u g'_{yz} - \rho\left(\cos\theta_u\right) g'_{xy} - \rho\left(\sin\theta_u\right) g'_{yy} \\ B'_z - \rho\left(\cos\theta_u\right) g'_{xz} - \rho\left(\sin\theta_u\right) g'_{yz} + L_u g'_{zz} \end{pmatrix}
\end{aligned} \tag{48}
$$

in which

$$\mathbf{B}' \triangleq \begin{pmatrix} B'_x \\ B'_y \\ B'_z \end{pmatrix} \tag{49}$$

is the magnetic field at the origin of C'.

E. SQUID outputs.

In the instrument the gradiometer signal is produced by forming the sum

$$U = U_+ + U_- \tag{50}$$

where U_+ and U_- are the responses of the SQUIDs when operated in a flux-locked loop. In response to the magnetic fields in equations 47 and 48 U_+ and U_- may be written

$$U_+ = \mathbf{S}_1' \cdot \mathbf{B}_+' \left(\mathbf{r}_+' \right) + a_+ \qquad (51)$$

$$U_- = \mathbf{S}' \cdot \mathbf{B}_-' \left(\mathbf{r}_-' \right) + a_- \qquad (52)$$

where the $\mathbf{S}_1'$ and $\mathbf{S}'$ denote the SQUID sensitivity vectors. These are normal to the SQUIDs with lengths proportional to the sensitivity of the SQUID system (V / T) and the a are arbitrary offset voltages which can change value upon each reset of the SQUID system electronics. In a perfectly aligned system $\mathbf{S}_1' = S_1 \mathbf{u}'$ and $\mathbf{S}' = -S\,\mathbf{u}'$ but in any real system the SQUID vectors will be tilted with respect to $\mathbf{u}'$ due to alignment errors. To characterize the alignment we introduce 2 pairs of alignment error angles. Let $\sigma_1 \left(\sigma \right)$ denote the error angle between the projection of the $\mathbf{S}_1' \left(\mathbf{S}' \right)$

vector onto the x'-y' plane and the unit vectors $\mathbf{u}'$ $(-\mathbf{u}')$, and τ_1 (τ) is the angle of inclination of $\mathbf{S}'_1$ $(\mathbf{S}')$ above the x'-y' plane. Then

$$\mathbf{S}'_1 = \begin{pmatrix} S_1 \cos \tau_1 \cos \theta_1 \\ S_1 \cos \tau_1 \sin \theta_1 \\ S_1 \sin \tau_1 \end{pmatrix}, \ \theta_1 \triangleq (\theta_u + \sigma_1) \tag{53}$$

$$\mathbf{S}' = \begin{pmatrix} -S \cos \tau \cos \theta \\ -S \cos \tau \sin \theta \\ S \sin \tau \end{pmatrix}, \ \theta \triangleq (\theta_u + \sigma) \tag{54}$$

If $\sigma_1 = \sigma = \tau_1 = \tau = 0$ then $\mathbf{S}'_1 = S_1 \mathbf{u}'$ and $\mathbf{S}' = -S \mathbf{u}'$ as required.

To simplify these introduce

$$\left. \begin{array}{l} M_1 \triangleq S_1 \cos \tau_1, \ N_1 \triangleq S_1 \sin \tau_1 \\ M \triangleq S \cos \tau , \ N \triangleq S \sin \tau \end{array} \right\} \tag{55}$$

so that 53 and 54 become

$$\mathbf{S}'_1 = \begin{pmatrix} M_1 \cos \theta_1 \\ M_1 \sin \theta_1 \\ N_1 \end{pmatrix} \tag{56}$$

$$\mathbf{S}' = \begin{pmatrix} -M \cos \theta \\ -M \sin \theta \\ N \end{pmatrix} \tag{57}$$

Using 47 and 56 in equation 51 gives

$$U_+ = \begin{pmatrix} M_1 \cos \theta_1 \\ M_1 \sin \theta_1 \\ N_1 \end{pmatrix} \cdot \begin{pmatrix} B'_x + L_u g'_{xz} + \rho (\sin \theta_u) g'_{xy} + \rho (\cos \theta_u) g'_{xx} \\ B'_y + L_u g'_{yz} + \rho (\cos \theta_u) g'_{xy} + \rho (\sin \theta_u) g'_{yy} \\ B'_z + \rho (\cos \theta_u) g'_{xz} + \rho (\sin \theta_u) g'_{yz} + L_u g'_{zz} \end{pmatrix} + a_+ \tag{58}$$

$$= (M_1 \cos \theta_1) \left(B'_x + L_u g'_{xz} + \rho (\sin \theta_u) g'_{xy} + \rho (\cos \theta_u) g'_{xx} \right) \tag{59}$$

$$+ (M_1 \sin \theta_1) \left(B'_y + L_u g'_{yz} + \rho (\cos \theta_u) g'_{xy} + \rho (\sin \theta_u) g'_{yy} \right) \tag{60}$$

$$+ (N_1) \left(B'_z + \rho (\cos \theta_u) g'_{xz} + \rho (\sin \theta_u) g'_{yz} + L_u g'_{zz} \right) + a_+ \tag{61}$$

$$= B'_x M_1 \cos \theta_1 + L_u g'_{xz} M_1 \cos \theta_1 + g'_{xx} \rho M_1 (\cos \theta_u) (\cos \theta_1) + g'_{xy} \rho M_1 (\sin \theta_u) (\cos \theta_1)$$

$$+ B'_y M_1 \sin \theta_1 + g'_{yz} L_u M_1 \sin \theta_1 + g'_{yy} \rho M_1 (\sin \theta_u) (\sin \theta_1) + g'_{xy} \rho M_1 (\cos \theta_u) (\sin \theta_1)$$

$$+ B'_z N_1 + g'_{zz} N_1 L_u + g'_{xz} N_1 \rho (\cos \theta_u) + g'_{yz} N_1 \rho (\sin \theta_u) + a_+ \tag{62}$$

where we have used that $g_{xy} = g_{yx}$, $g_{xz} = g_{zx}$ and $g_{yz} = g_{zy}$ from our knowledge of the properties of the magnetic gradient tensor.

Similarly using 48 and 57 in equation 52 gives

$$U_- = \begin{pmatrix} -M \cos\theta \\ -M \sin\theta \\ N \end{pmatrix} \cdot \begin{pmatrix} B'_x + L_u g'_{xz} - \rho \left(\sin\theta_u\right) g'_{xy} - \rho \left(\cos\theta_u\right) g'_{xx} \\ B'_y + L_u g'_{yz} - \rho \left(\cos\theta_u\right) g'_{xy} - \rho \left(\sin\theta_u\right) g'_{yy} \\ B'_z - \rho \left(\cos\theta_u\right) g'_{xz} - \rho \left(\sin\theta_u\right) g'_{yz} + L_u g'_{zz} \end{pmatrix} + a_- \tag{63}$$

$$= \left(-M \cos\theta\right)\left(B'_x + L_u g'_{xz} - \rho \left(\sin\theta_u\right) g'_{xy} - \rho \left(\cos\theta_u\right) g'_{xx}\right) \tag{64}$$

$$+ \left(-M \sin\theta\right)\left(B'_y + L_u g'_{yz} - \rho \left(\cos\theta_u\right) g'_{xy} - \rho \left(\sin\theta_u\right) g'_{yy}\right) \tag{65}$$

$$+ \left(N\right)\left(B'_z - \rho \left(\cos\theta_u\right) g'_{xz} - \rho \left(\sin\theta_u\right) g'_{yz} + L_u g'_{zz}\right) + a_- \tag{66}$$

$$= -B'_x M \cos\theta - L_u g'_{xz} M \cos\theta + M \rho g'_{xx} \left(\cos\theta_u\right)\left(\cos\theta\right) + M \rho g'_{xy} \left(\sin\theta_u\right)\left(\cos\theta\right)$$

$$- M B'_y \left(\sin\theta\right) - g'_{yz} M L_u \left(\sin\theta\right) + g'_{yy} M \rho \left(\sin\theta_u\right)\left(\sin\theta\right) + g'_{xy} M \rho \left(\cos\theta_u\right)\left(\sin\theta\right)$$

$$+ B'_z N + g'_{zz} N L_u - g'_{xz} N \rho \left(\cos\theta_u\right) - g'_{yz} N \rho \left(\sin\theta_u\right) + a_- \tag{67}$$

Substituting 62 and 67 into 50 gives the gradiometer output

$$U = B'_x M_1 \cos\theta_1 + L_u g'_{xz} M_1 \cos\theta_1 + g'_{xx}\rho M_1 (\cos\theta_u)(\cos\theta_1) + g'_{xy}\rho M_1 (\sin\theta_u)(\cos\theta_1)$$
$$- B'_x M\ \cos\theta\ - L_u g'_{xz} M\ \cos\theta\ + M\ \rho g'_{xy}(\sin\theta_u)(\cos\theta\ ) + M\ \rho g'_{xx}(\cos\theta_u)(\cos\theta\ )$$
$$+ B'_y M_1 \sin\theta_1 + g'_{yz} L_u M_1 \sin\theta_1 + g'_{yy}\rho M_1 (\sin\theta_u)(\sin\theta_1) + g'_{xy}\rho M_1 (\cos\theta_u)(\sin\theta_1)$$
$$- M\ B'_y (\sin\theta\ ) - g'_{yz} M\ L_u (\sin\theta\ ) + g'_{xy} M\ \rho(\cos\theta_u)(\sin\theta\ ) + g'_{yy} M\ \rho(\sin\theta_u)(\sin\theta\ )$$
$$+ B'_z N_1 + g'_{zz} N_1 L_u + g'_{xz} N_1 \rho(\cos\theta_u) + g'_{yz} N_1 \rho(\sin\theta_u) + a_+$$
$$+ B'_z N\ + g'_{zz} N\ L_u - g'_{xz} N\ \rho(\cos\theta_u) - g'_{yz} N\ \rho(\sin\theta_u) + a_-$$

$$= B'_x M_1 \cos(\theta_u + \sigma_1) + L_u g'_{xz} M_1 \cos(\theta_u + \sigma_1) + g'_{xx}\rho M_1 (\cos\theta_u)(\cos(\theta_u + \sigma_1))$$
$$+ g'_{xy}\rho M_1 (\sin\theta_u)(\cos(\theta_u + \sigma_1)) - B'_x M\ \cos(\theta_u + \sigma\ ) - L_u g'_{xz} M\ \cos(\theta_u + \dot\sigma\ )$$
$$+ M\ \rho g'_{xy}(\sin\theta_u)(\cos(\theta_u + \sigma\ )) + M\ \rho g'_{xx}(\cos\theta_u)(\cos(\theta_u + \sigma\ ))$$
$$+ B'_y M_1 \sin(\theta_u + \sigma_1) + g'_{yz} L_u M_1 \sin(\theta_u + \sigma_1) + g'_{yy}\rho M_1 (\sin\theta_u)(\sin(\theta_u + \sigma_1))$$
$$+ g'_{xy}\rho M_1 (\cos\theta_u)(\sin(\theta_u + \sigma_1)) - M\ B'_y (\sin(\theta_u + \sigma\ ))$$
$$- g'_{yz} M\ L_u (\sin(\theta_u + \sigma\ )) + g'_{xy} M\ \rho(\cos\theta_u)(\sin(\theta_u + \sigma\ ))$$
$$+ g'_{yy} M\ \rho(\sin\theta_u)(\sin(\theta_u + \sigma\ )) + B'_z N_1 + g'_{zz} N_1 L_u + g'_{xz} \dot N_1 \rho(\cos\theta_u)$$
$$+ g'_{yz} N_1 \rho(\sin\theta_u) + a_+ + B'_z N\ + g'_{zz} N\ L_u$$
$$- g'_{xz} N\ \rho(\cos\theta_u) - g'_{yz} N\ \rho(\sin\theta_u) + a_-$$

$$= B'_x M_1 (\cos\sigma_1 \cos\theta_u - \sin\sigma_1 \sin\theta_u) + L_u g'_{xz} M_1 (\cos\sigma_1 \cos\theta_u - \sin\sigma_1 \sin\theta_u)$$
$$+ g'_{xx}\rho M_1 (\cos\sigma_1 \cos\ \theta_u - \sin\sigma_1 \cos\theta_u \sin\theta_u) + g'_{xy}\rho M_1 (\cos\sigma_1 \cos\theta_u \sin\theta_u - \sin\sigma_1 \sin\ \theta_u)$$
$$- B'_x M\ (\cos\sigma\ \cos\theta_u - \sin\sigma\ \sin\theta_u) - L_u g'_{xz} M\ (\cos\sigma\ \cos\theta_u - \sin\sigma\ \sin\theta_u)$$
$$+ M\ \rho g'_{xy}(\cos\sigma\ \cos\theta_u \sin\theta_u - \sin\sigma\ \sin\ \theta_u) + M\ \rho g'_{xx}(\cos\sigma\ \cos\ \theta_u - \sin\sigma\ \cos\theta_u \sin\theta_u)$$
$$+ B'_y M_1 (\cos\sigma_1 \sin\theta_u + \sin\sigma_1 \cos\theta_u) + g'_{yz} L_u M_1 (\cos\sigma_1 \sin\theta_u + \sin\sigma_1 \cos\theta_u)$$
$$+ g'_{yy}\rho M_1 (\sin\sigma_1 \cos\theta_u \sin\theta_u + \cos\sigma_1 \sin\ \theta_u) + g'_{xy}\rho M_1 (\cos\sigma_1 \cos\theta_u \sin\theta_u + \sin\sigma_1 \cos\ \theta_u)$$
$$- M\ B'_y (\cos\sigma\ \sin\theta_u + \sin\sigma\ \cos\theta_u) - g'_{yz} M\ L_u (\cos\sigma\ \sin\theta_u + \sin\sigma\ \cos\theta_u)$$
$$+ g'_{xy} M\ \rho(\cos\sigma\ \cos\theta_u \sin\theta_u + \sin\sigma\ \cos\ \theta_u) + g'_{yy} M\ \rho(\sin\sigma\ \cos\theta_u \sin\theta_u + \cos\sigma\ \sin\ \theta_u)$$
$$+ B'_z N_1 + g'_{zz} N_1 L_u + g'_{xz} N_1 \rho(\cos\theta_u) + g'_{yz} N_1 \rho(\sin\theta_u) + a_+$$
$$+ B'_z N\ + g'_{zz} N\ L_u - g'_{xz} N\ \rho(\cos\theta_u) - g'_{yz} N\ \rho(\sin\theta_u) + a_-$$

$$= B'_x M_1 \cos \sigma_1 \cos \theta_u - B'_x M_1 \sin \sigma_1 \sin \theta_u + L_u g'_{xz} M_1 \cos \sigma_1 \cos \theta_u$$

$$- L_u g'_{xz} M_1 \sin \sigma_1 \sin \theta_u + g'_{xx} \rho M_1 \cos \sigma_1 \cos \theta_u - g'_{xx} \rho M_1 \sin \sigma_1 \cos \theta_u \sin \theta_u$$

$$+ g'_{xy} \rho M_1 \cos \sigma_1 \cos \theta_u \sin \theta_u - g'_{xy} \rho M_1 \sin \sigma_1 \sin \theta_u - B'_x M \cos \sigma \cos \theta_u$$

$$+ B'_x M \sin \sigma \sin \theta_u - L_u g'_{xz} M \cos \sigma \cos \theta_u + L_u g'_{xz} M \sin \sigma \sin \theta_u$$

$$+ M \rho g'_{xy} \cos \sigma \cos \theta_u \sin \theta_u - M \rho g'_{xy} \sin \sigma \sin \theta_u + M \rho g'_{xx} \cos \sigma \cos \theta_u$$

$$- M \rho g'_{xx} \sin \sigma \cos \theta_u \sin \theta_u + B'_y M_1 \cos \sigma_1 \sin \theta_u + B'_y M_1 \sin \sigma_1 \cos \theta_u$$

$$+ g'_{yz} L_u M_1 \cos \sigma_1 \sin \theta_u + g'_{yz} L_u M_1 \sin \sigma_1 \cos \theta_u + g'_{yy} \rho M_1 \sin \sigma_1 \cos \theta_u \sin \theta_u$$

$$+ g'_{yy} \rho M_1 \cos \sigma_1 \sin \theta_u + g'_{xy} \rho M_1 \cos \sigma_1 \cos \theta_u \sin \theta_u + g'_{xy} \rho M_1 \sin \sigma_1 \cos \theta_u$$

$$- M B'_y \cos \sigma \sin \theta_u - M B'_y \sin \sigma \cos \theta_u - g'_{yz} M L_u \cos \sigma \sin \theta_u$$

$$- g'_{yz} M L_u \sin \sigma \cos \theta_u + g'_{xy} M \rho \cos \sigma \cos \theta_u \sin \theta_u + g'_{xy} M \rho \sin \sigma \cos \theta_u$$

$$+ g'_{yy} M \rho \sin \sigma \cos \theta_u \sin \theta_u + g'_{yy} M \rho \cos \sigma \sin \theta_u$$

$$+ B'_z N_1 + g'_{zz} N_1 L_u + g'_{xz} N_1 \rho (\cos \theta_u) + g'_{yz} N_1 \rho (\sin \theta_u) + a_+$$

$$+ B'_z N + g'_{zz} N L_u - g'_{xz} N \rho (\cos \theta_u) - g'_{yz} N \rho (\sin \theta_u) + a_-$$

$$= B'_x M_1 \cos\theta_u \cos\sigma_1 - B'_x M \cos\theta_u \cos\sigma$$

$$+ B'_x M \sin\sigma \sin\theta_u - B'_x M_1 \sin\sigma_1 \sin\theta_u$$

$$+ B'_y M_1 \cos\sigma_1 \sin\theta_u - B'_y M \cos\sigma \sin\theta_u$$

$$+ B'_y M_1 \sin\sigma_1 \cos\theta_u - B'_y M \sin\sigma \cos\theta_u$$

$$+ L_u g'_{xz} M_1 \cos\sigma_1 \cos\theta_u - L_u g'_{xz} M \cos\sigma \cos\theta_u$$

$$- L_u g'_{xz} M_1 \sin\sigma_1 \sin\theta_u + L_u g'_{xz} M \sin\sigma \sin\theta_u$$

$$+ g'_{yz} L_u M_1 \cos\sigma_1 \sin\theta_u - g'_{yz} M L_u \cos\sigma \sin\theta_u$$

$$+ g'_{yz} L_u M_1 \sin\sigma_1 \cos\theta_u - g'_{yz} M L_u \sin\sigma \cos\theta_u$$

$$+ B'_z N_1 + B'_z N + g'_{zz} N_1 L_u + g'_{zz} N L_u + a_+ + a_-$$

$$+ g'_{xx} \rho M_1 \cos\sigma_1 \cos\theta_u + M \rho g'_{xx} \cos\sigma \cos\theta_u$$

$$+ g'_{yy} \rho M_1 \cos\sigma_1 \sin\theta_u + g'_{yy} M \rho \cos\sigma \sin\theta_u$$

$$+ g'_{xy} \rho M_1 \sin\sigma_1 \cos\theta_u + g'_{xy} M \rho \sin\sigma \cos\theta_u$$

$$- g'_{xy} \rho M_1 \sin\sigma_1 \sin\theta_u - M \rho g'_{xy} \sin\sigma \sin\theta_u$$

$$- g'_{xx} \rho M_1 \sin\sigma_1 \cos\theta_u \sin\theta_u - M \cdot \rho g'_{xx} \sin\sigma \cos\theta_u \sin\theta_u$$

$$+ g'_{yy} \rho M_1 \sin\sigma_1 \cos\theta_u \sin\theta_u + g'_{yy} M \rho \sin\sigma \cos\theta_u \sin\theta_u$$

$$+ 2 g'_{xy} \rho M_1 \cos\sigma_1 \cos\theta_u \sin\theta_u + 2M \rho g'_{xy} \cos\sigma \cos\theta_u \sin\theta_u$$

$$+ g'_{xz} N_1 \rho (\cos\theta_u) - g'_{xz} N \rho (\cos\theta_u)$$

$$+ g'_{yz} N_1 \rho (\sin\theta_u) - g'_{yz} N \rho (\sin\theta_u)$$

$$= (M_1 \cos \sigma_1 - M \cos \sigma) B'_x \cos \theta_u$$

$$+ (M \sin \sigma - M_1 \sin \sigma_1) B'_x \sin \theta_u$$

$$+ (M_1 \cos \sigma_1 - M \cos \sigma) B'_y \sin \theta_u$$

$$+ (M_1 \sin \sigma_1 - M \sin \sigma) B'_y \cos \theta_u$$

$$+ L_u (M_1 \cos \sigma_1 - M \cos \sigma) g'_{xz} \cos \theta_u$$

$$- L_u (M_1 \sin \sigma_1 - M \sin \sigma) g'_{xz} \sin \theta_u$$

$$+ L_u (M_1 \cos \sigma_1 - M \cos \sigma) g'_{yz} \sin \theta_u$$

$$+ L_u (M_1 \sin \sigma_1 - M \sin \sigma) g'_{yz} \cos \theta_u$$

$$+ L_u (N_1 + N) g'_{zz}$$

$$+ B'_z (N_1 + N)$$

$$+ a$$

$$+ \rho (M_1 \cos \sigma_1 + M \cos \sigma) g'_x (\cos \theta_u)$$

$$+ \rho (M_1 \cos \sigma_1 + M \cos \sigma) g'_y (\sin \theta_u)$$

$$+ \rho (M_1 \sin \sigma_1 + M \sin \sigma) g'_{xy} (\cos \theta_u)$$

$$- \rho (M_1 \sin \sigma_1 + M \sin \sigma) g'_{xy} (\sin \theta_u)$$

$$- \rho (M_1 \sin \sigma_1 + M \sin \sigma) g'_x (\sin \theta_u) (\cos \theta_u)$$

$$+ \rho (M_1 \sin \sigma_1 + M \sin \sigma) g'_y (\sin \theta_u) (\cos \theta_u)$$

$$+ 2\rho (M_1 \cos \sigma_1 + M \cos \sigma) g'_{xy} (\sin \theta_u) (\cos \theta_u)$$

$$+ \rho g'_{xz} (\cos \theta_u) (N_1 - N)$$

$$+ \rho g'_{yz} (\sin \theta_u) (N_1 - N)$$

where $a \triangleq a_+ + a_-$. Expanding the trigonometric functions within each term and collecting

harmonics gives

$$U = \left(L_u g'_{zz} + B'_z\right)(N_1 + N\ ) + a$$
$$+ \frac{1}{2}\rho\left(M_1 \cos\sigma_1 + M\ \cos\sigma\ \right)\left(g'_{xx} + g'_{yy}\right)$$

$$+ \left(M_1 \cos\sigma_1 - M\ \cos\sigma\ \right)\left(B'_x + L_u g'_{xz}\right)\cos\theta_u$$
$$+ \left(M_1 \sin\sigma_1 - M\ \sin\sigma\ \right)\left(B'_y + L_u g'_{yz}\right)\cos\theta_u$$
$$+ \rho\left(N_1 - N\ \right) g'_{xz}\cos\theta_u$$

$$- \left(M_1 \sin\sigma_1 - M\ \sin\sigma\ \right)\left(B'_x + L_u g'_{xz}\right)\sin\theta_u$$
$$+ \left(M_1 \cos\sigma_1 - \dot{M}\ \cos\sigma\ \right)\left(B'_y + L_u g'_{yz}\right)\sin\theta_u$$
$$+ \rho\left(N_1 - N\ \right) g'_{yz}\sin\theta_u$$

$$+ \frac{1}{2}\rho\left(M_1 \cos\sigma_1 + M\ \cos\sigma\ \right)\left(g'_{xx} - g'_{yy}\right)\cos 2\theta_u$$
$$+ \rho\left(M_1 \sin\sigma_1 + M\ \sin\sigma\ \right) g'_{xy}\cos 2\theta_u$$

$$+ \frac{1}{2}\rho\left(M_1 \sin\sigma_1 + M\ \sin\sigma\ \right)\left(g'_{yy} - g'_{xx}\right)\sin 2\theta_u$$
$$+ \rho\left(M_1 \cos\sigma_1 + M\ \cos\sigma\ \right) g'_{xy}\sin 2\theta_u. \tag{68}$$

Hence

$$U = \mu_u \left(L_u g'_{zz} + B'_z \right) + a$$
$$+ \xi_u \left(g'_{xx} + g'_{yy} \right)$$

$$+ \varsigma_u \left(B'_x + L_u g'_{xz} \right) \cos \theta_u$$
$$+ \eta_u \left(B'_y + L_u g'_{yz} \right) \cos \theta_u$$
$$+ v_u g'_{xz} \cos \theta_u$$

$$- \eta_u \left(B'_x + L_u g'_{xz} \right) \sin \theta_u$$
$$+ \varsigma_u \left(B'_y + L_u g'_{yz} \right) \sin \theta_u$$
$$+ v_u g'_{yz} \sin \theta_u$$

$$+ \xi_u \left(g'_{xx} - g'_{yy} \right) \cos 2\theta_u$$
$$+ 2 \left(\chi_u \right) g'_{xy} \cos 2\theta_u$$

$$+ \chi_u \left(g'_{yy} - g'_{xx} \right) \sin 2\theta_u$$
$$+ 2 \left(\xi_u \right) g'_{xy} \sin 2\theta_u. \tag{69}$$

where

$$\left.\begin{aligned}
\xi_u &\triangleq \tfrac{1}{2}\rho \left(M_1 \cos \sigma_1 + M \cos \sigma \right) \\
\chi_u &\triangleq \tfrac{1}{2}\rho \left(M_1 \sin \sigma_1 + M \sin \sigma \right) \\
v_u &\triangleq \rho \left(N_1 - N \right) \\
\varsigma_u &\triangleq \left(M_1 \cos \sigma_1 - M \cos \sigma \right) \\
\eta_u &\triangleq \left(M_1 \sin \sigma_1 - M \sin \sigma \right) \\
\mu_u &\triangleq \left(N_1 + N \right)
\end{aligned}\right\} \tag{70}$$

N.B. In the absence of alignment errors, i.e. $\sigma_1 = \sigma = 0$, $\tau_1 = \tau = 0$ from equation 55

$$\left.\begin{aligned}
M_1 &= S_1, \ N_1 = 0 \\
M &= S \ , \ N = 0
\end{aligned}\right\}$$

and from equations 70 χ_u, η_u, v_u and μ_u vanish, and

$$\left.\begin{aligned}
\xi_u &= \tfrac{1}{2}\rho \left(S_1 + S \right) \\
\varsigma_u &= \left(S_1 - S \right).
\end{aligned}\right\} \tag{71}$$

Finally if $S_1 = S \triangleq S$ say, then ς_u also vanishes and the only non-zero constant is

$$\xi_u = \rho S. \tag{72}$$

III. DATA EXTRACTION

One method to extract the data is to take a series of measurements over a number of angular increments of the pick-up loop structure and solve the resulting system of equations algebraically. A series of voltage measurements is now taken at angles $\theta_u = -\pi/4,\ 0,\ \pi/4,\ \pi/2,\ 3\pi/4,\ \pi,\ 5\pi/4,$

$3\pi/2$ to yield the following set of equations:

$$U_0 \triangleq U(\theta = -\pi/4) = a + B'_3\mu_u + g'_{11}\xi_u + g'_{11}\chi_u - 2g'_1\ \xi_u + g'\ \xi_u - g'\ \chi_u + L_u g'_{33}\mu_u$$
$$+ \tfrac{1}{2}B'_1\eta_u\sqrt{2} + \tfrac{1}{2}B'\eta_u\sqrt{2} + \tfrac{1}{2}B'_1\varsigma_u\sqrt{2} - \tfrac{1}{2}B'\varsigma_u\sqrt{2} + \tfrac{1}{2}g'_{13}\upsilon_u\sqrt{2} - \tfrac{1}{2}g'_3\upsilon_u\sqrt{2}$$
$$+ \tfrac{1}{2}L_u g'_{13}\eta_u\sqrt{2} + \tfrac{1}{2}L_u g'_3\eta_u\sqrt{2} + \tfrac{1}{2}L_u g'_{13}\varsigma_u\sqrt{2} - \tfrac{1}{2}L_u g'_3\varsigma_u\sqrt{2}$$

$$U \triangleq U(\theta = \pi/4) = a + B'_3\mu_u + g'_{11}\xi_u - g'_{11}\chi_u + 2g'_1\ \xi_u + g'\ \xi_u + g'\ \chi_u + L_u g'_{33}\mu_u$$
$$- \tfrac{1}{2}B'_1\eta_u\sqrt{2} + \tfrac{1}{2}B'\eta_u\sqrt{2} + \tfrac{1}{2}B'_1\varsigma_u\sqrt{2} + \tfrac{1}{2}B'\varsigma_u\sqrt{2} + \tfrac{1}{2}g'_{13}\upsilon_u\sqrt{2} + \tfrac{1}{2}g'_3\upsilon_u\sqrt{2}$$
$$- \tfrac{1}{2}L_u g'_{13}\eta_u\sqrt{2} + \tfrac{1}{2}L_u g'_3\eta_u\sqrt{2} + \tfrac{1}{2}L_u g'_{13}\varsigma_u\sqrt{2} + \tfrac{1}{2}L_u g'_3\varsigma_u\sqrt{2}$$

$$U_4 \triangleq U(\theta = 3\pi/4) = a + B'_3\mu_u + g'_{11}\xi_u + g'_{11}\chi_u - 2g'_1\ \xi_u + g'\ \xi_u - g'\ \chi_u + L_u g'_{33}\mu_u$$
$$- \tfrac{1}{2}B'_1\eta_u\sqrt{2} - \tfrac{1}{2}B'\eta_u\sqrt{2} - \tfrac{1}{2}B'_1\varsigma_u\sqrt{2} + \tfrac{1}{2}B'\varsigma_u\sqrt{2} - \tfrac{1}{2}g'_{13}\upsilon_u\sqrt{2} + \tfrac{1}{2}g'_3\upsilon_u\sqrt{2}$$
$$- \tfrac{1}{2}L_u g'_{13}\eta_u\sqrt{2} - \tfrac{1}{2}L_u g'_3\eta_u\sqrt{2} - \tfrac{1}{2}L_u g'_{13}\varsigma_u\sqrt{2} + \tfrac{1}{2}L_u g'_3\varsigma_u\sqrt{2}$$

$$U_6 \triangleq U(\theta = 5\pi/4) = a + B'_3\mu_u + g'_{11}\xi_u - g'_{11}\chi_u + 2g'_1\ \xi_u + g'\ \chi_u + L_u g'_{33}\mu_u$$
$$+ \tfrac{1}{2}B'_1\eta_u\sqrt{2} - \tfrac{1}{2}B'\eta_u\sqrt{2} - \tfrac{1}{2}B'_1\varsigma_u\sqrt{2} - \tfrac{1}{2}B'\varsigma_u\sqrt{2} - \tfrac{1}{2}g'_{13}\upsilon_u\sqrt{2} - \tfrac{1}{2}g'_3\upsilon_u\sqrt{2} \tag{73}$$
$$+ \tfrac{1}{2}L_u g'_{13}\eta_u\sqrt{2} - \tfrac{1}{2}L_u g'_3\eta_u\sqrt{2} - \tfrac{1}{2}L_u g'_{13}\varsigma_u\sqrt{2} - \tfrac{1}{2}L_u g'_3\varsigma_u\sqrt{2}$$

$$U_1 \triangleq U(\theta = 0) = a + B'\eta_u + B'_1\varsigma_u + B'_3\mu_u + 2g'_{11}\xi_u + g'_{13}\upsilon_u + 2g'_1\ \chi_u$$
$$+ L_u g'_3\eta_u + L_u g'_{13}\varsigma_u + L_u g'_{33}\mu_u$$

$$U_3 \triangleq U(\theta = \pi/2) = a - B'_1\eta_u + B'\varsigma_u + B'_3\mu_u - 2g'_1\ \chi_u + g'_3\upsilon_u + 2g'\ \xi_u$$
$$- L_u g'_{13}\eta_u + L_u g'_3\varsigma_u + L_u g'_{33}\mu_u$$

$$U_5 \triangleq U(\theta = \pi) = a - B'\eta_u - B'_1\varsigma_u + B'_3\mu_u + 2g'_{11}\xi_u - g'_{13}\upsilon_u + 2g'_1\ \chi_u$$
$$- L_u g'_3\eta_u - L_u g'_{13}\varsigma_u + L_u g'_{33}\mu_u$$

$$U_7 \triangleq U(\theta = 3\pi/2) = a + B'_1\eta_u - B'\varsigma_u + B'_3\mu_u - 2g'_1\ \chi_u - g'_3\upsilon_u + 2g'\ \xi_u$$
$$+ L_u g'_{13}\eta_u - L_u g'_3\varsigma_u + L_u g'_{33}\mu_u$$

Note that even with all of these misalignment errors considered the following L.C. vanishes if the SQUID stays in lock throughout the measurement series:

$$U_0 - U_1 + U\ - U_3 + U_4 - U_5 + U_6 - U_7 = 0 \tag{74}$$

So this is still a good check to test the SQUID lock and the integrity of the data with respect to

random pointing errors which might occur during the measurement sequence.

A. Solving for gradient components.

Assuming that ξ_u, χ_u, ς_u, η_u, ν_u, μ_u are all known by calibration, the 8 equations in 73 contain 10 unknown independent variables, namely B'_x, B'_y, B'_z, g'_{xx}, g'_{yy}, g'_{zz}, g'_{xy}, g'_{yz}, g'_{zx}, a, since voltages $U_0...U_7$ are determined by measurement. The set of equations in 73 is clearly underdetermined and additional data from the other two drums is necessary to determine a complete solution. Nevertheless it is possible to solve for the off-diagonal gradient component, g'_{xy}, and the difference between the diagonal components, $g'_{xx} - g'_{yy}$, independently of the measurements from the other two drums. This is done by forming the following linear combinations of eqs. 73 to eliminate the dependence on the SQUID offset voltage a and on the field components,

$$U_0 - U + U_4 - U_6 = 4\chi_u\Delta' - 8\xi_u g'_{xy} \tag{75}$$

$$U_1 - U_3 + U_5 - U_7 = 4\xi_u\Delta' + 8\chi_u g'_{xy}, \tag{76}$$

where

$$\Delta' \triangleq g'_{xx} - g'_{yy}.$$

Then

$$\begin{pmatrix} g'_{xy} \\ \Delta' \end{pmatrix} = \begin{pmatrix} \dfrac{\chi_u(U_1 - U_3 + U_5 - U_7) - \xi_u(U_0 - U + U_4 - U_6)}{8(\xi_u + \chi_u)} \\ \dfrac{\chi_u(U_0 - U + U_4 - U_6) + \xi_u(U_1 - U_3 + U_5 - U_7)}{4(\xi_u + \chi_u)} \end{pmatrix} \tag{77}$$

provided

$$(\xi_u + \chi_u) \neq 0 \tag{78}$$

which is always satisfied because both ξ_u and χ_u are real and $\xi_u > 0$.

The results from these additional gradiometers give solutions with exactly the same form in their respective coordinate systems, i.e.

$$\begin{pmatrix} g''_{xy} \\ \Delta'' \end{pmatrix} = \begin{pmatrix} \dfrac{\chi_v(V_1 - V_3 + V_5 - V_7) - \xi_v(V_0 - V + V_4 - V_6)}{8(\xi_v + \chi_v)} \\ \dfrac{\chi_v(V_0 - V + V_4 - V_6) + \xi_v(V_1 - V_3 + V_5 - V_7)}{4(\xi_v + \chi_v)} \end{pmatrix} \tag{79}$$

and

$$\begin{pmatrix} g'''_{xy} \\ \\ \Delta''' \end{pmatrix} = \begin{pmatrix} \dfrac{\chi_w \left(W_1 - W_3 + W_5 - W_7\right) - \xi_w \left(W_0 - W + W_4 - W_6\right)}{8 \left(\xi_w + \chi_w\right)} \\ \\ \dfrac{\chi_w \left(W_0 - W + W_4 - W_6\right) + \xi_w \left(W_1 - W_3 + W_5 - W_7\right)}{4 \left(\xi_w + \chi_w\right)} \end{pmatrix}$$ (80)

Equations 77, 79 and 80 are now transformed to the instrument coordinate system and the resulting system of equations may be solved to yield the gradient measurements.

1. Transformations for the "proof of concept" instrument.

In the case of the present embodiment $\lambda = \pi/4$, and the transformations for the xy, xx and yy gradient components of the C', C'' and C''' coordinate systems are those given by the relevant equations in 36, 37 and 44, which are brought together here for convenience:

$$\left. \begin{aligned} g'_1 &= \tfrac{1}{2}\sqrt{2}\left(g_1 - g_3\right) \\ g''_1 &= -\tfrac{1}{4}\sqrt{2}g_1 + \tfrac{1}{4}\sqrt{2}g_3 + \tfrac{1}{8}\sqrt{6}g_{11} + \tfrac{1}{4}\sqrt{6}g_{13} - \tfrac{1}{8}\sqrt{6}g \\ g'''_1 &= -\tfrac{1}{4}\sqrt{2}g_1 + \tfrac{1}{4}\sqrt{2}g_3 - \tfrac{1}{8}\sqrt{6}g_{11} - \tfrac{1}{4}\sqrt{6}g_{13} + \tfrac{1}{8}\sqrt{6}g \\ \\ g'_{11} &= \tfrac{1}{4}g_{11} - g_{13} + \tfrac{1}{4}g_{33} \\ g' &= g \\ g''_{11} &= \tfrac{1}{8}g_{11} + \tfrac{1}{4}g_{13} + \tfrac{3}{8}g + \tfrac{1}{4}g_{33} - \tfrac{1}{4}\sqrt{3}g_1 - \tfrac{1}{4}\sqrt{3}g_3 \\ g'' &= \tfrac{3}{4}g_{11} + \tfrac{1}{4}g + \tfrac{1}{4}g_1\sqrt{3} \\ g'''_{11} &= \tfrac{1}{8}g_{11} + \tfrac{1}{4}g_{13} + \tfrac{3}{8}g + \tfrac{1}{4}g_{33} + \tfrac{1}{4}\sqrt{3}g_1 + \tfrac{1}{4}\sqrt{3}g_3 \\ g''' &= \tfrac{3}{4}g_{11} + \tfrac{1}{4}g - \tfrac{1}{4}\sqrt{3}g_1 \end{aligned} \right\}$$ (81)

Using the last six of these equations we can derive the transformation equations for the diagonal component differences Δ', Δ'' and Δ''', i.e.

$$\begin{aligned} \Delta' &= g'_{11} - g' \\ &= \tfrac{1}{2}g_{11} - g_{13} - g + \tfrac{1}{2}g_{33} \\ &= \tfrac{1}{2}\Delta_1 - \tfrac{1}{2}\Delta - g_{13} \end{aligned}$$ (82)

$$\begin{aligned} \Delta'' &= g''_{11} - g'' \\ &= -\tfrac{5}{8}g_{11} + \tfrac{1}{8}g + \tfrac{1}{2}g_{33} - \tfrac{3}{4}g_1\sqrt{3} - \tfrac{1}{2}g_3\sqrt{3} + \tfrac{1}{2}g_{13} \\ &= -\tfrac{1}{8}\Delta_1 + \tfrac{1}{2}\Delta_3 - \tfrac{3}{4}g_1\sqrt{3} - \tfrac{1}{2}g_3\sqrt{3} + \tfrac{1}{2}g_{13} \end{aligned}$$ (83)

$$\Delta''' = g'''_{11} - g'''$$

$$= -\frac{5}{8}g_{11} + \frac{1}{8}g \ \ + \frac{1}{2}g_{33} + \frac{3}{4}g_1 \ \sqrt{3} + \frac{1}{2}g \ {}_3\sqrt{3} + \frac{1}{2}g_{13}$$

$$= -\frac{1}{8}\Delta_1 + \frac{1}{2}\Delta_3 + \frac{3}{4}g_1 \ \sqrt{3} + \frac{1}{2}g \ {}_3\sqrt{3} + \frac{1}{2}g_{13} \qquad (84)$$

where

$$\left.\begin{array}{l} \Delta_1 \triangleq g_{11} - g \\ \Delta \ \triangleq g \ \ - g_{33} \\ \Delta_3 \triangleq g_{33} - g_{11} \end{array}\right\} \qquad (85)$$

Combining these with the first three equation of 81 yields the following set of equations

$$\left.\begin{array}{l} g'_1 \ = \frac{1}{}\sqrt{2}g_1 \ - \frac{1}{}\sqrt{2}g \ {}_3 \\ g''_1 = -\frac{1}{4}\sqrt{2}g_1 \ + \frac{1}{4}\sqrt{2}g \ {}_3 + \frac{1}{4}\sqrt{6}g_{13} + \frac{1}{8}\sqrt{6}\Delta_1 \\ g'''_1 = -\frac{1}{4}\sqrt{2}g_1 \ + \frac{1}{4}\sqrt{2}g \ {}_3 - \frac{1}{4}\sqrt{6}g_{13} - \frac{1}{8}\sqrt{6}\Delta_1 \\ \Delta' = \frac{1}{}\Delta_1 - \frac{1}{}\Delta \ - g_{13} \\ \Delta'' = -\frac{1}{8}\Delta_1 + \frac{1}{}\Delta_3 - \frac{3}{4}\sqrt{3}g_1 \ - \frac{1}{}\sqrt{3}g \ {}_3 + \frac{1}{}g_{13} \\ \Delta''' = -\frac{1}{8}\Delta_1 + \frac{1}{}\Delta_3 + \frac{3}{4}\sqrt{3}g_1 \ + \frac{1}{}\sqrt{3}g \ {}_3 + \frac{1}{}g_{13} \end{array}\right\} \qquad (86)$$

Although the corresponding matrix of coefficients, with respect to the variables $g_1 \ , g \ {}_3, g_{13}, \Delta_1, \Delta \ , \Delta_3,$

$$\begin{pmatrix} -\frac{1}{}\sqrt{2} & \frac{1}{}\sqrt{2} & 0 & 0 & 0 & 0 \\ \frac{1}{4}\sqrt{2} & -\frac{1}{4}\sqrt{2} & -\frac{1}{4}\sqrt{6} & -\frac{1}{8}\sqrt{6} & 0 & 0 \\ \frac{1}{4}\sqrt{2} & -\frac{1}{4}\sqrt{2} & \frac{1}{4}\sqrt{6} & \frac{1}{8}\sqrt{6} & 0 & 0 \\ 0 & 0 & 1 & -\frac{1}{} & \frac{1}{} & 0 \\ \frac{3}{4}\sqrt{3} & \frac{1}{}\sqrt{3} & -\frac{1}{} & \frac{1}{8} & 0 & -\frac{1}{} \\ -\frac{3}{4}\sqrt{3} & -\frac{1}{}\sqrt{3} & -\frac{1}{} & \frac{1}{8} & 0 & -\frac{1}{} \end{pmatrix} \begin{pmatrix} g_1 \\ g \ {}_3 \\ g_{13} \\ \Delta_1 \\ \Delta \\ \Delta_3 \end{pmatrix} = \begin{pmatrix} -g'_1 \\ -g'_1 \\ -g'_1 \\ -\Delta'_1 \\ -\Delta' \\ -\Delta'_3 \end{pmatrix}$$

is singular a solution can by obtained after observing that, by definition,

$$\Delta_1 + \Delta \ + \Delta_3 = g_{11} - g \ + g \ - g_{33} + g_{33} - g_{11} = 0$$

and use this equation to augment 86, i.e.

$$
\begin{aligned}
g_1' &= \tfrac{1}{2}\sqrt{2}\,g_1 - \tfrac{1}{2}\sqrt{2}\,g_3 \\
g_1'' &= -\tfrac{1}{4}\sqrt{2}\,g_1 + \tfrac{1}{4}\sqrt{2}\,g_3 + \tfrac{1}{4}\sqrt{6}\,g_{13} + \tfrac{1}{8}\sqrt{6}\,\Delta_1 \\
g_1''' &= -\tfrac{1}{4}\sqrt{2}\,g_1 + \tfrac{1}{4}\sqrt{2}\,g_3 - \tfrac{1}{4}\sqrt{6}\,g_{13} - \tfrac{1}{8}\sqrt{6}\,\Delta_1 \\
\Delta' &= \tfrac{1}{2}\Delta_1 - \tfrac{1}{2}\Delta - g_{13} \\
\Delta'' &= -\tfrac{1}{8}\Delta_1 + \tfrac{1}{2}\Delta_3 - \tfrac{3}{4}\sqrt{3}\,g_1 - \tfrac{1}{2}\sqrt{3}\,g_3 + \tfrac{1}{2}g_{13} \\
\Delta''' &= -\tfrac{1}{8}\Delta_1 + \tfrac{1}{2}\Delta_3 + \tfrac{3}{4}\sqrt{3}\,g_1 + \tfrac{1}{2}\sqrt{3}\,g_3 + \tfrac{1}{2}g_{13} \\
0 &= \Delta_1 + \Delta + \Delta_3
\end{aligned}
\left. \right\}
$$

The corresponding matrix form

$$
\begin{pmatrix}
-\tfrac{1}{2}\sqrt{2} & \tfrac{1}{2}\sqrt{2} & 0 & 0 & 0 & 0 \\
\tfrac{1}{4}\sqrt{2} & -\tfrac{1}{4}\sqrt{2} & -\tfrac{1}{4}\sqrt{6} & -\tfrac{1}{8}\sqrt{6} & 0 & 0 \\
\tfrac{1}{4}\sqrt{2} & -\tfrac{1}{4}\sqrt{2} & \tfrac{1}{4}\sqrt{6} & \tfrac{1}{8}\sqrt{6} & 0 & 0 \\
0 & 0 & 1 & -\tfrac{1}{2} & \tfrac{1}{2} & 0 \\
\tfrac{3}{4}\sqrt{3} & \tfrac{1}{2}\sqrt{3} & -\tfrac{1}{2} & \tfrac{1}{8} & 0 & -\tfrac{1}{2} \\
-\tfrac{3}{4}\sqrt{3} & -\tfrac{1}{2}\sqrt{3} & -\tfrac{1}{2} & \tfrac{1}{8} & 0 & -\tfrac{1}{2} \\
0 & 0 & 0 & -1 & -1 & -1
\end{pmatrix}
\begin{pmatrix}
g_1 \\ g_3 \\ g_{13} \\ \Delta_1 \\ \Delta \\ \Delta_3
\end{pmatrix}
=
\begin{pmatrix}
-g_1' \\ -g_1'' \\ -g_1''' \\ -\Delta' \\ -\Delta'' \\ -\Delta''' \\ 0
\end{pmatrix}
$$

is transformed to a square matrix by multiplying on the left by the transpose of the coefficient matrix to give

$$
\begin{pmatrix}
\tfrac{33}{8} & \tfrac{3}{2} & 0 & 0 & 0 & 0 \\
\tfrac{3}{2} & \tfrac{9}{4} & 0 & 0 & 0 & 0 \\
0 & 0 & \tfrac{9}{4} & -\tfrac{1}{4} & \tfrac{1}{2} & 1 \\
0 & 0 & -\tfrac{1}{4} & \tfrac{47}{3} & \tfrac{3}{4} & \tfrac{7}{8} \\
0 & 0 & \tfrac{1}{2} & \tfrac{3}{4} & \tfrac{5}{4} & 1 \\
0 & 0 & 1 & \tfrac{7}{8} & 1 & \tfrac{3}{2}
\end{pmatrix}
\begin{pmatrix}
g_1 \\ g_3 \\ g_{13} \\ \Delta_1 \\ \Delta \\ \Delta_3
\end{pmatrix}
=
\begin{pmatrix}
\tfrac{1}{2}g_1'\sqrt{2} - \tfrac{1}{4}g_1''\sqrt{2} - \tfrac{3}{4}\Delta''\sqrt{3} - \tfrac{1}{4}g_1'''\sqrt{2} + \tfrac{3}{4}\Delta'''\sqrt{3} \\
-\tfrac{1}{2}g_1'\sqrt{2} + \tfrac{1}{4}g_1''\sqrt{2} - \tfrac{1}{2}\Delta''\sqrt{3} + \tfrac{1}{4}g_1'''\sqrt{2} + \tfrac{1}{2}\Delta'''\sqrt{3} \\
-\Delta' + \tfrac{1}{2}\Delta'' + \tfrac{1}{2}\Delta''' + \tfrac{1}{4}g_1''\sqrt{6} - \tfrac{1}{4}g_1'''\sqrt{6} \\
\tfrac{1}{2}\Delta' - \tfrac{1}{8}\Delta'' - \tfrac{1}{8}\Delta''' + \tfrac{1}{8}g_1''\sqrt{6} - \tfrac{1}{8}g_1'''\sqrt{6} \\
-\tfrac{1}{2}\Delta' \\
\tfrac{1}{2}\Delta'' + \tfrac{1}{2}\Delta'''
\end{pmatrix}
$$

which is now non-singular. The inverse of the coefficient matrix is

$$
\begin{pmatrix}
\tfrac{8}{5} & -\tfrac{16}{75} & 0 & 0 & 0 & 0 \\
-\tfrac{16}{75} & \tfrac{44}{75} & 0 & 0 & 0 & 0 \\
0 & 0 & \tfrac{136}{5} & \tfrac{88}{5} & -\tfrac{64}{5} & -\tfrac{6}{5} \\
0 & 0 & \tfrac{88}{5} & \tfrac{304}{5} & -\tfrac{11}{5} & -\tfrac{44}{75} \\
0 & 0 & -\tfrac{64}{5} & -\tfrac{11}{5} & \tfrac{436}{5} & -\tfrac{68}{75} \\
0 & 0 & -\tfrac{6}{5} & -\tfrac{44}{75} & -\tfrac{68}{75} & \tfrac{17}{75}
\end{pmatrix}
$$

and the solution of the system of equations, 86, is

$$
\begin{pmatrix} g_1 \\ g_3 \\ g_{13} \\ \Delta_1 \\ \Delta \\ \Delta_3 \end{pmatrix} = \begin{pmatrix} -\frac{\sqrt{3}}{15}(\Delta''' - \Delta'') + \frac{\sqrt{}}{15}(2g_1' - g_1'' - g_1''') \\ -\frac{\sqrt{3}}{15}(\Delta''' - \Delta'') + \frac{\sqrt{}}{5}(g_1'' - 2g_1' + g_1''') \\ \frac{}{15}(\Delta'' - 2\Delta' + \Delta''') + \frac{\sqrt{6}}{5}(g_1'' - g_1''') \\ \frac{8}{15}\Delta' - \frac{4}{15}\Delta'' - \frac{4}{15}\Delta''' + \frac{4}{15}g_1''\sqrt{6} - \frac{4}{15}g_1'''\sqrt{6} \\ -\frac{14}{15}\Delta' - \frac{8}{15}\Delta'' - \frac{8}{15}\Delta''' - \frac{}{15}g_1''\sqrt{6} + \frac{}{15}g_1'''\sqrt{6} \\ \frac{}{5}\Delta' + \frac{4}{5}\Delta'' + \frac{4}{5}\Delta''' - \frac{}{15}g_1''\sqrt{6} + \frac{}{15}g_1'''\sqrt{6} \end{pmatrix} \tag{87}
$$

Note that at this point this solution has been obtained without using the tracelessness of the gradient tensor and is written in terms of the differences between the diagonal components. The diagonal components, however, follow immediately from equations 85 and the traceless of the gradient tensor, i.e. given that

$$
\left. \begin{array}{l} g_{11} - g = \Delta_1 \\ g - g_{33} = \Delta \\ g_{33} - g_{11} = \Delta_3 \\ g_{11} + g + g_{33} = 0 \end{array} \right\} \Rightarrow \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} g_{11} \\ g \\ g_{33} \end{pmatrix} = \begin{pmatrix} \Delta_1 \\ \Delta \\ \Delta_3 \\ 0 \end{pmatrix} \tag{88}
$$

then

$$
\begin{pmatrix} g_{11} \\ g \\ g_{33} \end{pmatrix} = \frac{1}{3}\begin{pmatrix} -\Delta_3 + \Delta_1 \\ -\Delta_1 + \Delta \\ -\Delta + \Delta_3 \end{pmatrix} \tag{89}
$$

Substituting for the component differences from the last three equations in 87 gives

$$
\begin{pmatrix} g_{11} \\ g \\ g_{33} \end{pmatrix} = \begin{pmatrix} -\frac{}{45}(-\Delta' + 8\Delta'' + 8\Delta''') + \frac{\sqrt{6}}{15}(g_1'' - g_1''') \\ -\frac{}{45}(11\Delta' + 2\Delta'' + 2\Delta''') + \frac{\sqrt{6}}{15}(g_1''' - g_1'') \\ \frac{4}{9}(\Delta' + \Delta'' + \Delta''') \end{pmatrix} \tag{90}
$$

The final solution for the gradient components follow after substituting for the Δ's and g's from equations 77, 79 and 80 into equation 90 and the first three equations of 87. The resulting

equations are brought together here for convenience.

$$g_{11} = -\tfrac{}{45}\left(-\Delta' + 8\Delta'' + 8\Delta'''\right) + \tfrac{\sqrt{6}}{15}\left(g_1'' - g_1'''\right)$$

$$g\ \ = -\tfrac{}{45}\left(11\Delta' + 2\Delta'' + 2\Delta'''\right) + \tfrac{\sqrt{6}}{15}\left(g_1''' - g_1''\right)$$

$$g_{33} = \tfrac{4}{9}\left(\Delta' + \Delta'' + \Delta'''\right)$$

$$g_1\ = \tfrac{\sqrt{3}}{15}\left(\Delta''' - \Delta''\right) + \tfrac{\sqrt{}}{15}\left(2g_1' - g_1'' - g_1'''\right)$$

$$g\ _3 = \tfrac{\sqrt{3}}{15}\left(\Delta''' - \Delta''\right) + \tfrac{\sqrt{}}{5}\left(g_1'' - 2g_1' + g_1'''\right)$$

$$g_{13} = \tfrac{}{15}\left(\Delta'' - 2\Delta' + \Delta'''\right) + \tfrac{\sqrt{6}}{5}\left(g_1'' - g_1'''\right)$$

in which

$$g_1' = \frac{\chi_u\left(U_1 - U_3 + U_5 - U_7\right) - \xi_u\left(U_0 - U\ + U_4 - U_6\right)}{8\left(\xi_u + \chi_u\right)}$$

$$g_1'' = \frac{\chi_v\left(V_1 - V_3 + V_5 - V_7\right) - \xi_v\left(V_0 - V\ + V_4 - V_6\right)}{8\left(\xi_v + \chi_v\right)} \tag{91}$$

$$g_1''' = \frac{\chi_w\left(W_1 - W_3 + W_5 - W_7\right) - \xi_w\left(W_0 - W\ + W_4 - W_6\right)}{8\left(\xi_w + \chi_w\right)}$$

$$\Delta' = \frac{\chi_u\left(U_0 - U\ + U_4 - U_6\right) + \xi_u\left(U_1 - U_3 + U_5 - U_7\right)}{4\left(\xi_u + \chi_u\right)}$$

$$\Delta'' = \frac{\chi_v\left(V_0 - V\ + V_4 - V_6\right) + \xi_v\left(V_1 - V_3 + V_5 - V_7\right)}{4\left(\xi_v + \chi_v\right)}$$

$$\Delta''' = \frac{\chi_w\left(W_0 - W\ + W_4 - W_6\right) + \xi_w\left(W_1 - W_3 + W_5 - W_7\right)}{4\left(\xi_w + \chi_w\right)}$$

and

$$\xi_u \triangleq \tfrac{1}{}\rho\left(M_1\cos\sigma_1 + M\ \cos\sigma\ \right)$$
$$\chi_u \triangleq \tfrac{1}{}\rho\left(M_1\sin\sigma_1 + M\ \sin\sigma\ \right)$$
$$M_1 \triangleq S_1\cos\tau_1, \quad N_1 \triangleq S_1\sin\tau_1$$
$$M\ \triangleq S\ \cos\tau\ , \quad N\ \triangleq S\ \sin\tau$$

2. *Special case: "Proof of concept instrument" with zero alignment errors.*

If all of the alignment errors vanish then

$$\xi_u = {}^1\!\rho\,(S_1 + S\;)$$
$$\chi_u = 0$$

and equation 91 reduces to

$$g_{11} = -\tfrac{}{45}\left(-\Delta' + 8\Delta'' + 8\Delta'''\right) + \tfrac{\sqrt{6}}{15}\left(g_1'' - g_1'''\right)$$

$$g\quad = -\tfrac{}{45}\left(11\Delta' + 2\Delta'' + 2\Delta'''\right) + \tfrac{\sqrt{6}}{15}\left(g_1''' - g_1''\right)$$

$$g_{33} = \tfrac{4}{9}\left(\Delta' + \Delta'' + \Delta'''\right)$$

$$g_1\;\; = \tfrac{\sqrt{3}}{15}\left(\Delta''' - \Delta''\right) + \tfrac{\sqrt{}}{15}\left(2g_1' - g_1'' - g_1'''\right)$$

$$g\;_3 = \tfrac{\sqrt{3}}{15}\left(\Delta''' - \Delta''\right) + \tfrac{\sqrt{}}{5}\left(g_1'' - 2g_1' + g_1'''\right)$$

$$g_{13} = \tfrac{}{15}\left(\Delta'' - 2\Delta' + \Delta'''\right) + \tfrac{\sqrt{6}}{5}\left(g_1'' - g_1'''\right)$$

and   (92)

$$g_1' = \frac{-(U_0 - U\; + U_4 - U_6)}{8\xi_u}$$

$$g_1'' = \frac{-(V_0 - V\; + V_4 - V_6)}{8\xi_v}$$

$$g_1''' = \frac{-(W_0 - W\; + W_4 - W_6)}{8\xi_w}$$

$$\Delta' = \frac{(U_1 - U_3 + U_5 - U_7)}{4\xi_u}$$

$$\Delta'' = \frac{(V_1 - V_3 + V_5 - V_7)}{4\xi_v}$$

$$\Delta''' = \frac{(W_1 - W_3 + W_5 - W_7)}{4\xi_w}$$

IV. SOLVING FOR FIELD COMPONENTS.

If the gradiometer is intentionally unbalanced, i.e. $S_1 \neq S$, then it is capable of returning information about the magnetic field in addition to the gradient. As for the gradient case considered above we form linear combinations of 73 to both eliminate the dependence on the SQUID offset voltage, a, and to reduce the number of gradient terms. To eliminate the SQUID offset we start with the differences between voltages corresponding to opposing positions, and then take the sums and differences of these pairs, i.e.

$$(U_0 - U_4) + (U - U_6) = 2\eta_u B' \sqrt{2} + 2\varsigma_u B_1' \sqrt{2} + 2\upsilon_u g_{13}' \sqrt{2} + 2L_u \eta_u g'_3 \sqrt{2} + 2L_u \varsigma_u g_{13}' \sqrt{2}$$

$$(U_0 - U_4) - (U - U_6) = 2\eta_u B_1' \sqrt{2} - 2\varsigma_u B' \sqrt{2} - 2\upsilon_u g'_3 \sqrt{2} + 2L_u \eta_u g_{13}' \sqrt{2} - 2L_u \varsigma_u g'_3 \sqrt{2}$$

$$(U_1 - U_5) + (U_3 - U_7) = 2B' \eta_u - 2B_1' \eta_u + 2B_1' \varsigma_u + 2B' \varsigma_u$$
$$+ 2g_{13}' \upsilon_u + 2g'_3 \upsilon_u - 2L_u g_{13}' \eta_u + 2L_u g'_3 \eta_u + 2L_u g_{13}' \varsigma_u + 2L_u g'_3 \varsigma_u$$

$$(U_1 - U_5) - (U_3 - U_7) = 2B_1' \eta_u + 2B' \eta_u + 2B_1' \varsigma_u - 2B' \varsigma_u$$
$$+ 2g_{13}' \upsilon_u - 2g'_3 \upsilon_u + 2L_u g_{13}' \eta_u + 2L_u g'_3 \eta_u + 2L_u g_{13}' \varsigma_u - 2L_u g'_3 \varsigma_u$$

and then simplify further by forming the following L.C. of these equations:

$$\frac{U_1 + U_3 - U_5 - U_7}{2} - \frac{U_0 + U - U_4 - U_6}{2\sqrt{2}} = \varsigma_u B' - \eta_u B_1' + \upsilon_u g'_3 - L_u \eta_u g_{13}' + L_u \varsigma_u g'_3 \qquad (93)$$

$$\frac{U_1 - U_3 - U_5 + U_7}{2} - \frac{U_0 - U - U_4 + U_6}{2\sqrt{2}} = \eta_u B' + \varsigma_u B_1' + \upsilon_u g_{13}' + L_u \eta_u g'_3 + L_u \varsigma_u g_{13}' \qquad (94)$$

As for the gradient calculation previously the results from the other two drums have exactly the same form in their respective coordinate systems, i.e.

$$\frac{V_1 + V_3 - V_5 - V_7}{2} - \frac{V_0 + V - V_4 - V_6}{2\sqrt{2}} = \varsigma_v B'' - \eta_v B_1'' + \upsilon_v g''_3 - L_v \eta_v g_{13}'' + L_v \varsigma_v g''_3 \qquad (95)$$

$$\frac{V_1 - V_3 - V_5 + V_7}{2} - \frac{V_0 - V - V_4 + V_6}{2\sqrt{2}} = \eta_v B'' + \varsigma_v B_1'' + \upsilon_v g_{13}'' + L_v \eta_v g''_3 + L_v \varsigma_v g_{13}'' \qquad (96)$$

$$\frac{W_1 + W_3 - W_5 - W_7}{2} - \frac{W_0 + W - W_4 - W_6}{2\sqrt{2}} = \varsigma_w B''' - \eta_w B_1'''$$
$$+ \upsilon_w g'''_3 - L_w \eta_w g_{13}''' + L_w \varsigma_w g'''_3 \qquad (97)$$

$$\frac{W_1 - W_3 - W_5 + W_7}{2} - \frac{W_0 - W\ - W_4 + W_6}{2\sqrt{2}} = \eta_w B''' + \varsigma_w B_1'''$$
$$+\ v_w g'''_3 + L_w \eta_w g'''_3 + L_w \varsigma_w g'''_{13} \qquad (98)$$

Equations 93 - ?? may be simplified by making the following substitutions:

$$\left.\begin{aligned}
U^\dagger &\triangleq \frac{U_1 + U_3 - U_5 - U_7}{2} - \frac{U_0 + U\ - U_4 - U_6}{2\sqrt{2}} - g'_3 (L_u \varsigma_u + v_u) + L_u g'_{13} \eta_u \\
U^\ddagger &\triangleq \frac{U_1 - U_3 - U_5 + U_7}{2} - \frac{U_0 - U\ - U_4 + U_6}{2\sqrt{2}} - g'_{13} (L_u \varsigma_u + v_u) - L_u g'_3 \eta_u \\
V^\dagger &\triangleq \frac{V_1 + V_3 - V_5 - V_7}{2} - \frac{V_0 + V\ - V_4 - V_6}{2\sqrt{2}} + L_v g''_{13} \eta_v - g''_3 v_v - L_v g''_3 \varsigma_v \\
V^\ddagger &\triangleq \frac{V_1 - V_3 - V_5 + V_7}{2} - \frac{V_0 - V\ - V_4 + V_6}{2\sqrt{2}} - g''_{13} v_v - L_v g''_3 \eta_v - L_v g''_{13} \varsigma_v \\
W^\dagger &\triangleq \frac{W_1 + W_3 - W_5 - W_7}{2} - \frac{W_0 + W\ - W_4 - W_6}{2\sqrt{2}} + L_w g'''_{13} \eta_w - g'''_3 v_w - L_w g'''_3 \varsigma_w \\
W^\ddagger &\triangleq \frac{W_1 - W_3 - W_5 + W_7}{2} - \frac{W_0 - W\ - W_4 + W_6}{2\sqrt{2}} - g'''_3 v_w - L_w g'''_3 \eta_w - L_w g'''_{13} \varsigma_w
\end{aligned}\right\} \qquad (99)$$

Observe that all of the terms on the right hand sides of these equations are known i.e. the U, V and W voltages are returned by instrument, the gradient components are known in the instrument frame and can be easily found using coordinare transformations, and the remaining terms are known by calibration. With these substitutions 93 - ?? become

$$\left.\begin{aligned}
U^\dagger &= \varsigma_u B' - \eta_u B_1' \\
U^\ddagger &= \eta_u B' + \varsigma_u B_1' \\
V^\dagger &= \varsigma_v B'' - \eta_v B_1'' \\
V^\ddagger &= \eta_v B'' + \varsigma_v B_1'' \\
W^\dagger &= \varsigma_w B''' - \eta_w B_1''' \\
W^\ddagger &= \eta_w B''' + \varsigma_w B_1'''
\end{aligned}\right\} \qquad (100)$$

Equations 100, corresponding to measurements in the C', C'' and C''' coordinate systems, may be solved by transforming to the common instrument coordinate system using the coordinate transformations for field components derived earlier, but the resulting sets of equations rapidly

become intractible. It is simpler to first solve these pairwise in their respective coordinate systems,

$$\begin{pmatrix} B'_1 \\ B' \end{pmatrix} = \frac{1}{\eta_u + \varsigma_u} \begin{pmatrix} U^{\ddagger}\varsigma_u - U^{\dagger}\eta_u \\ U^{\ddagger}\eta_u + U^{\dagger}\varsigma_u \end{pmatrix}$$ (101)

$$\begin{pmatrix} B''_1 \\ B'' \end{pmatrix} = \frac{1}{\eta_v + \varsigma_v} \begin{pmatrix} V^{\ddagger}\varsigma_v - V^{\dagger}\eta_v \\ V^{\ddagger}\eta_v + V^{\dagger}\varsigma_v \end{pmatrix}$$ (102)

$$\begin{pmatrix} B'''_1 \\ B''' \end{pmatrix} = \frac{1}{\eta_w + \varsigma_w} \begin{pmatrix} W^{\ddagger}\varsigma_w - W^{\dagger}\eta_w \\ W^{\ddagger}\eta_w + W^{\dagger}\varsigma_w \end{pmatrix}$$ (103)

and then carry out the transformation to the instrument system. As derived above the relevant transformation equations for the field components are

$$\mathbf{B}' = \begin{pmatrix} \sin\lambda & 0 & -\cos\lambda \\ 0 & 1 & 0 \\ \cos\lambda & 0 & \sin\lambda \end{pmatrix} \mathbf{B}$$

$$\Rightarrow \begin{cases} B'_1 = -(\cos\lambda)\,B_3 + (\sin\lambda)\,B_1 \\ B' = B \\ B'_3 = (\cos\lambda)\,B_1 + (\sin\lambda)\,B_3 \end{cases}$$

$$\mathbf{B}'' = \begin{pmatrix} -\tfrac{1}{2}\sin\lambda & \tfrac{1}{2}\sqrt{3}\,(\sin\lambda) & -\cos\lambda \\ -\tfrac{1}{2}\sqrt{3} & -\tfrac{1}{2} & 0 \\ -\tfrac{1}{2}\cos\lambda & \tfrac{1}{2}\sqrt{3}\,(\cos\lambda) & \sin\lambda \end{pmatrix} \mathbf{B}$$

$$\Rightarrow \begin{cases} B''_1 = -(\cos\lambda)\,B_3 - \tfrac{1}{2}(\sin\lambda)\,B_1 + \tfrac{1}{2}(\sin\lambda)\,B\cdot\sqrt{3} \\ B'' = -\tfrac{1}{2}B - \tfrac{1}{2}B_1\sqrt{3} \\ B''_3 = -\tfrac{1}{2}(\cos\lambda)\,B_1 + (\sin\lambda)\,B_3 + \tfrac{1}{2}(\cos\lambda)\,B\,\sqrt{3} \end{cases}$$

$$\mathbf{B}''' = \begin{pmatrix} -\tfrac{1}{2}\sin\lambda & -\tfrac{1}{2}\sqrt{3}\,(\sin\lambda) & -\cos\lambda \\ \tfrac{1}{2}\sqrt{3}\cdot & -\tfrac{1}{2} & 0 \\ -\tfrac{1}{2}\cos\lambda & -\tfrac{1}{2}\sqrt{3}\,(\cos\lambda) & \sin\lambda \end{pmatrix} \mathbf{B}$$

$$\Rightarrow \begin{cases} B'''_1 = -(\cos\lambda)\,B_3 - \tfrac{1}{2}(\sin\lambda)\,B_1 - \tfrac{1}{2}(\sin\lambda)\,B\,\sqrt{3} \\ B''' = -\tfrac{1}{2}B + \tfrac{1}{2}B_1\sqrt{3} \\ B'''_3 = -\tfrac{1}{2}(\cos\lambda)\,B_1 + (\sin\lambda)\,B_3 - \tfrac{1}{2}(\cos\lambda)\,B\,\sqrt{3} \end{cases}$$

and if $\lambda = \pi/4$

$$
\left.
\begin{aligned}
B_1' &= \tfrac{1}{}\sqrt{2}\,(B_1 - B_3) \\
B' &= B \\
B_1'' &= \tfrac{1}{4}\sqrt{2}\,(-B_1 + \sqrt{3}B - 2B_3) \\
B'' &= -\tfrac{1}{}\,(B_1\sqrt{3} + B\,) \\
B_1''' &= \tfrac{1}{4}\sqrt{2}\,(-B_1 - \sqrt{3}B - 2B_3) \\
B''' &= \tfrac{1}{}\,(\sqrt{3}B_1 - B\,)
\end{aligned}
\right\}
\tag{104}
$$

and equations 101 - 103 become

$$
\left.
\begin{aligned}
\frac{U^{\ddagger}\varsigma_u - U^{\dagger}\eta_u}{\eta_u + \varsigma_u} &= \tfrac{1}{}\sqrt{2}\,(B_1 - B_3) \\[2mm]
\frac{U^{\ddagger}\eta_u + U^{\dagger}\varsigma_u}{\eta_u + \varsigma_u} &= B \\[2mm]
\frac{V^{\ddagger}\varsigma_v - V^{\dagger}\eta_v}{\eta_v + \varsigma_v} &= \tfrac{1}{4}\sqrt{2}\,(-B_1 + \sqrt{3}B - 2B_3) \\[2mm]
\frac{V^{\ddagger}\eta_v + V^{\dagger}\varsigma_v}{\eta_v + \varsigma_v} &= -\tfrac{1}{}\,(B_1\sqrt{3} + B\,) \\[2mm]
\frac{W^{\ddagger}\varsigma_w - W^{\dagger}\eta_w}{\eta_w + \varsigma_w} &= \tfrac{1}{4}\sqrt{2}\,(-B_1 - \sqrt{3}B - 2B_3) \\[2mm]
\frac{W^{\ddagger}\eta_w + W^{\dagger}\varsigma_w}{\eta_w + \varsigma_w} &= \tfrac{1}{}\,(\sqrt{3}B_1 - B\,)
\end{aligned}
\right\}
\tag{105}
$$

A least-squares best-fit solution may be found by first rewriting this overdetermined system of equations in matrix form in terms of the unknowns B_1, B, B_3 to give

$$
\begin{pmatrix}
\tfrac{1}{}\sqrt{2} & 0 & -\tfrac{1}{}\sqrt{2} \\
0 & 1 & 0 \\
-\tfrac{1}{4}\sqrt{2} & \tfrac{1}{4}\sqrt{6} & -\tfrac{1}{}\sqrt{2} \\
-\tfrac{1}{}\sqrt{3} & -\tfrac{1}{} & 0 \\
-\tfrac{1}{4}\sqrt{2} & -\tfrac{1}{4}\sqrt{6} & -\tfrac{1}{}\sqrt{2} \\
\tfrac{1}{}\sqrt{3} & -\tfrac{1}{} & 0
\end{pmatrix}
\begin{pmatrix}
B_1 \\ B \\ B_3
\end{pmatrix}
=
\begin{pmatrix}
\dfrac{U^{\ddagger}\varsigma_u - U^{\dagger}\eta_u}{\eta_u + \varsigma_u} \\[2mm]
\dfrac{U^{\ddagger}\eta_u + U^{\dagger}\varsigma_u}{\eta_u + \varsigma_u} \\[2mm]
\dfrac{V^{\ddagger}\varsigma_v - V^{\dagger}\eta_v}{\eta_v + \varsigma_v} \\[2mm]
\dfrac{V^{\ddagger}\eta_v + V^{\dagger}\varsigma_v}{\eta_v + \varsigma_v} \\[2mm]
\dfrac{W^{\ddagger}\varsigma_w - W^{\dagger}\eta_w}{\eta_w + \varsigma_w} \\[2mm]
\dfrac{W^{\ddagger}\eta_w + W^{\dagger}\varsigma_w}{\eta_w + \varsigma_w}
\end{pmatrix}
$$

and then multiplying on the left by the transpose of the coefficient matrix to yield

$$
\begin{pmatrix} \frac{9}{4}B_1 \\ \frac{9}{4}B \\ \frac{3}{4}B_3 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}\sqrt{3}\dfrac{V^{\ddagger}\eta_v + V^{\dagger}\varsigma_v}{\eta_v + \varsigma_v} + \frac{1}{2}\sqrt{3}\dfrac{W^{\ddagger}\eta_w + W^{\dagger}\varsigma_w}{\eta_w + \varsigma_w} \\ \quad +\frac{1}{2}\sqrt{2}\dfrac{U^{\ddagger}\varsigma_u - U^{\dagger}\eta_u}{\eta_u + \varsigma_u} - \frac{1}{4}\sqrt{2}\dfrac{V^{\ddagger}\varsigma_v - V^{\dagger}\eta_v}{\eta_v + \varsigma_v} - \frac{1}{4}\sqrt{2}\dfrac{W^{\ddagger}\varsigma_w - W^{\dagger}\eta_w}{\eta_w + \varsigma_w} \\[2ex] \dfrac{U^{\ddagger}\eta_u + U^{\dagger}\varsigma_u}{\eta_u + \varsigma_u} + \frac{1}{4}\sqrt{6}\dfrac{V^{\ddagger}\varsigma_v - V^{\dagger}\eta_v}{\eta_v + \varsigma_v} \\ \quad -\frac{1}{4}\sqrt{6}\dfrac{W^{\ddagger}\varsigma_w - W^{\dagger}\eta_w}{\eta_w + \varsigma_w} - \frac{1}{2}\dfrac{V^{\ddagger}\eta_v + V^{\dagger}\varsigma_v}{\eta_v + \varsigma_v} - \frac{1}{2}\dfrac{W^{\ddagger}\eta_w + W^{\dagger}\varsigma_w}{\eta_w + \varsigma_w} \\[2ex] -\frac{1}{2}\sqrt{2}\left(\dfrac{U^{\ddagger}\varsigma_u - U^{\dagger}\eta_u}{\eta_u + \varsigma_u} + \dfrac{V^{\ddagger}\varsigma_v - V^{\dagger}\eta_v}{\eta_v + \varsigma_v} + \dfrac{W^{\ddagger}\varsigma_w - W^{\dagger}\eta_w}{\eta_w + \varsigma_w} \right) \end{pmatrix} \qquad (106)
$$

We bring the final forms of the solutions here for convenience

$$B_1 = \frac{2\sqrt{2}}{9}\frac{U^{\ddagger}\varsigma_u - U^{\dagger}\eta_u}{\eta_u + \varsigma_u} - \frac{\sqrt{2}}{9}\frac{V^{\ddagger}\varsigma_v - V^{\dagger}\eta_v}{\eta_v + \varsigma_v} - \frac{2\sqrt{3}}{9}\frac{\left(V^{\ddagger}\eta_v + V^{\dagger}\varsigma_v\right)}{\eta_v + \varsigma_v}$$
$$- \frac{\sqrt{2}}{9}\frac{W^{\ddagger}\varsigma_w - W^{\dagger}\eta_w}{\eta_w + \varsigma_w} + \frac{2\sqrt{3}}{9}\frac{\left(W^{\ddagger}\eta_w + W^{\dagger}\varsigma_w\right)}{\eta_w + \varsigma_w} \tag{107}$$

$$B = \frac{4}{9}\frac{U^{\ddagger}\eta_u + U^{\dagger}\varsigma_u}{\eta_u + \varsigma_u} + \frac{\sqrt{6}}{9}\frac{V^{\ddagger}\varsigma_v - V^{\dagger}\eta_v}{\eta_v + \varsigma_v} - \frac{2}{9}\frac{V^{\ddagger}\eta_v + V^{\dagger}\varsigma_v}{\eta_v + \varsigma_v}$$
$$- \frac{\sqrt{6}}{9}\frac{W^{\ddagger}\varsigma_w - W^{\dagger}\eta_w}{\eta_w + \varsigma_w} - \frac{2}{9}\frac{W^{\ddagger}\eta_w + W^{\dagger}\varsigma_w}{\eta_w + \varsigma_w} \tag{108}$$

$$B_3 = -\frac{\sqrt{2}}{3}\left(\frac{U^{\ddagger}\varsigma_u - U^{\dagger}\eta_u}{\eta_u + \varsigma_u} + \frac{V^{\ddagger}\varsigma_v - V^{\dagger}\eta_v}{\eta_v + \varsigma_v} + \frac{W^{\ddagger}\varsigma_w - W^{\dagger}\eta_w}{\eta_w + \varsigma_w}\right) \tag{109}$$

where

$$\left.\begin{aligned}
U^{\dagger} &\triangleq \frac{U_1 + U_3 - U_5 - U_7}{2} - \frac{U_0 + U - U_4 - U_6}{2\sqrt{2}} + L_u g'_{13}\eta_u - g'_3 v_u - L_u g'_3 \varsigma_u \\
U^{\ddagger} &\triangleq \frac{U_1 - U_3 - U_5 + U_7}{2} - \frac{U_0 - U - U_4 + U_6}{2\sqrt{2}} - g'_{13} v_u - L_u g'_3 \eta_u - L_u g'_{13}\varsigma_u \\
V^{\dagger} &\triangleq \frac{V_1 + V_3 - V_5 - V_7}{2} - \frac{V_0 + V - V_4 - V_6}{2\sqrt{2}} + L_v g''_{13}\eta_v - g''_3 v_v - L_v g''_3 \varsigma_v \\
V^{\ddagger} &\triangleq \frac{V_1 - V_3 - V_5 + V_7}{2} - \frac{V_0 - V - V_4 + V_6}{2\sqrt{2}} - g''_{13} v_v - L_v g''_3 \eta_v - L_v g''_{13}\varsigma_v \\
W^{\dagger} &\triangleq \frac{W_1 + W_3 - W_5 - W_7}{2} - \frac{W_0 + W - W_4 - W_6}{2\sqrt{2}} + L_w g'''_{13}\eta_w - g'''_3 v_w - L_w g'''_3 \varsigma_w \\
W^{\ddagger} &\triangleq \frac{W_1 - W_3 - W_5 + W_7}{2} - \frac{W_0 - W - W_4 + W_6}{2\sqrt{2}} - g'''_3 v_w - L_w g'''_3 \eta_w - L_w g'''_{13}\varsigma_w
\end{aligned}\right\},$$

$$\begin{aligned}
g'_{13} &= \tfrac{1}{}g_{11} - \tfrac{1}{}g_{33}, & g''_3 &= -\tfrac{1}{4}\sqrt{2}g_1 - \tfrac{1}{4}\sqrt{2}g_3 + \tfrac{1}{8}\sqrt{6}g_{11} - \tfrac{1}{4}\sqrt{6}g_{13} - \tfrac{1}{8}\sqrt{6}g \\
g'_3 &= \tfrac{1}{}\sqrt{2}\left(g_1 + g_3\right), & g'''_{13} &= \tfrac{1}{8}g_{11} + \tfrac{3}{8}g - \tfrac{1}{}g_{33} + \tfrac{1}{4}\sqrt{3}g_1 \\
g''_{13} &= \tfrac{1}{8}g_{11} + \tfrac{3}{8}g - \tfrac{1}{}g_{33} - \tfrac{1}{4}\sqrt{3}g_1, & g'''_3 &= -\tfrac{1}{4}\sqrt{2}g_1 - \tfrac{1}{4}\sqrt{2}g_3 - \tfrac{1}{8}\sqrt{6}g_{11} + \tfrac{1}{4}\sqrt{6}g_{13} + \tfrac{1}{8}\sqrt{6}g
\end{aligned}$$

Formally the scaling coefficients, v_u, ς_u and η_u are defined by equations 55 and 70 but would normally be determined by calibration.

V. EXAMPLES

A. Precession about the z-axis.

Say $\sigma_1 = \sigma = 0$, $S_1 = S = S$. If the gradiometer precesses about the the axis of rotation by virtue of an error angle of ϕ say, then the theory developed above suffices to model this by letting $\tau_1 = -\tau = \phi$.[1]. Using equations ??

$$
\left.
\begin{aligned}
\xi_u &\triangleq \rho Q S \cos \phi \\
\chi_u &\triangleq 0 \\
\upsilon_u &\triangleq 2\rho Q S \sin \phi \\
\varsigma_u &\triangleq 0 \\
\eta_u &\triangleq 0
\end{aligned}
\right\}
\qquad
\begin{aligned}
\alpha_u &\triangleq \left(2Q \ S \ \rho \ \cos \ \phi\right)^{-1} \\
\beta_u &\triangleq (0)^{-1} \\
U^\dagger &\triangleq U_0 - U + U_4 - U_6 \\
U^\ddagger &\triangleq U_1 - U_3 + U_5 - U_7 \\
M_1 &\triangleq S \cos \phi, \ N_1 \triangleq S \sin \phi \\
M &\triangleq S \cos \phi, \ N \triangleq -S \sin \phi
\end{aligned}
\tag{110}
$$

VI. APPENDIX 1: GEOGRAPHIC COORDINATE SYSTEM, C_G.

The geographical coordinates system, C_G, is defined as follows:

x_G-axis: The x_G-axis is along the traverse in the direction of increasing station number ("grid north").

y_G-axis: The y_G-axis is at 90° to the x_G-axis, parallel to the ground and points right as you look along the traverse ("grid east").

z_G-axis: The z_G-axis is vertically down (pointing at the centre of mass of the earth).

This is a right-handed Cartesian coordinate system in terms of which a unit vector with *azimuth* A (the angle measured clockwise from grid north) and *inclination* I (angle measured positive downward from the horizontal) is

$$
\text{unit vector with } azimuth \ A \text{ and } inclination \ I = (\cos A \cos I, \sin A \cos I, \sin I). \tag{111}
$$

---

[1] If the gradiometer drum is tilted through and an angle ϕ in the x-z plane say, the pick-up loop vectors are also tilted through the same angle and the positions of the pick-up loops are altered. The change in position has a negligable effect on measurements of gradients produced by distance sources. Since by definition τ is the angle of the sensitivity vector above the x-y plane, a tilt though ϕ implies that $\tau_1 = \phi$ and $\tau = -\phi$.

1. Transformations between C and C_G.

The transformation between C and C_G assumes a particularly simple form if the instrument is oriented with its base horizontal and its x-axis aligned due north. In this case C is mapped to C_G by a positive rotation through an angle π radians about the x-axis and the transformations of the field and tensor and data become simply

$$\mathbf{B}_G = \widehat{R}_\pi \mathbf{B} \tag{112}$$

$$\widehat{g}_G = \widehat{R}_\pi \widehat{g} \widehat{R}_\pi^{-1} \tag{113}$$

where

$$\widehat{R}_\pi = \widehat{R}_\pi^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{114}$$

Hence

$$\begin{pmatrix} B_{G1} \\ B_G \\ B_{G3} \end{pmatrix} = \begin{pmatrix} B_1 \\ -B \\ -B_3 \end{pmatrix} \tag{115}$$

and

$$\begin{pmatrix} g_{G11} & g_{G1} & g_{G13} \\ g_{G\,1} & g_G & g_{G\,3} \\ g_{G31} & g_{G3} & g_{G33} \end{pmatrix} = \begin{pmatrix} g_{11} & -g_1 & -g_{13} \\ -g_1 & g & g_3 \\ -g_{31} & g_3 & g_{33} \end{pmatrix} \tag{116}$$

More generally if the instrument has azimuth A and inclination I (in geographic coordinates) then C is mapped to C_G by first rotating about y through an angle of $-I$ then by a rotation about the resulting z through an angle A followed lastly by a rotation about the resulting x of π. Hence

$$\mathbf{B}_G = \widehat{R}_x(\pi) \widehat{R}_z(A) \widehat{R}_y(-I) \mathbf{B} \tag{117}$$

$$\begin{pmatrix} B_{G1} \\ B_G \\ B_{G3} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\pi & \sin\pi \\ 0 & -\sin\pi & \cos\pi \end{pmatrix} \begin{pmatrix} \cos A & \sin A & 0 \\ -\sin A & \cos A & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos(-I) & 0 & -\sin(-I) \\ 0 & 1 & 0 \\ \sin(-I) & 0 & \cos(-I) \end{pmatrix} \begin{pmatrix} B_1 \\ B \\ B_3 \end{pmatrix}$$

$$= \begin{pmatrix} \cos A \cos I & \sin A & \cos A \sin I \\ \sin A \cos I & -\cos A & \sin A \sin I \\ \sin I & 0 & -\cos I \end{pmatrix} \begin{pmatrix} B_1 \\ B \\ B_3 \end{pmatrix}. \tag{118}$$

and

$$
\begin{pmatrix}
g_{G11} & g_{G1} & g_{G13} \\
g_{G1} & g_G & g_{G\,3} \\
g_{G13} & g_{G\,3} & g_{G33}
\end{pmatrix}
$$

$$
= \begin{pmatrix}
\cos A \cos I & \sin A & \cos A \sin I \\
\sin A \cos I & -\cos A & \sin A \sin I \\
\sin I & 0 & -\cos I
\end{pmatrix}
\begin{pmatrix}
g_{11} & g_1 & g_{13} \\
g_1 & g & g_{\,3} \\
g_{13} & g_{\,3} & g_{33}
\end{pmatrix}
\begin{pmatrix}
\cos A \cos I & \sin A \cos I & \sin I \\
\sin A & -\cos A & 0 \\
\cos A \sin I & \sin A \sin I & -\cos I
\end{pmatrix}
$$

In the special case $A = I = 0$ we get so that

$$
\begin{pmatrix}
B_{G1} \\
B_G \\
B_{G3}
\end{pmatrix}
=
\begin{pmatrix}
B_1 \\
-B \\
-B_3
\end{pmatrix}
$$

$$
\begin{pmatrix}
g_{G11} & g_{G1} & g_{G13} \\
g_{G1} & g_G & g_{G\,3} \\
g_{G13} & g_{G\,3} & g_{G33}
\end{pmatrix}
=
\begin{pmatrix}
g_{11} & -g_1 & -g_{13} \\
-g_1 & g & g_{\,3} \\
-g_{13} & g_{\,3} & g_{33}
\end{pmatrix}
$$

as above.

The approximation used in obtaining equation (3) above, which is significantly simplified as compared to equation (2), involves truncating the Taylor series expansion of the magnetic field as a function of position to include the effects of the B-field and the first-order gradients only. This simplifying approximation is desirable because the Taylor series expansion has an infinite number of terms. The approximation is justified because the magnitudes of the higher-order terms decreases as the order of the terms increases, for three reasons. The higher-order terms in the Taylor series are scaled by prefactors which may be written as the product of two factors. Firstly, one of these factors is a scaling term which decreases with increasing order. The second factor is proportional to $1/r^n$, where r is half the distance of separation between the field detectors, and n is the order of the term. Since the separation between the field sensors is small, raising this to an increasing power rapidly decreases the magnitude of this factor. The third reason the approximation is justified is that higher-order gradients fall off faster with distance from the field source than lower-order gradients, so the signal at the gradiometer is dominated by the field and the first-order gradient components.

The effects of misalignments of the field detectors on the output voltages from the instrument are also scaled by these higher-order terms, but these effects are small and similarly decrease with increasing order.

Referring to equations (6)-(8), it is noted that the value of λ may be any value in the range of 0° to 90°, inclusive. As noted in equation (9), when λ is approximately 35.3°, an orthogonal system is provided. In the presently described embodiment, λ is 45°.

It will be appreciated by persons skilled in the art that numerous variations and/or modifications may be made to the invention as shown in the specific embodiments without departing from the spirit or scope of the invention as broadly described. In particular, the present invention may be applied in conjunction with the invention set out in the co-pending PCT application claiming priority from Australian Provisional Patent Application No. 2002950624 and filed simultaneously with the present application, the contents of which are herein incorporated by reference. The present embodiments are, therefore, to be considered in all respects as illustrative and not restrictive.

CLAIMS

1.   A method of field detection comprising:

         providing a gradiometer in a field, the gradiometer having at least first and second field vector sensors connected in a differencing arrangement; and

5   controllably altering a position of the at least first and second field vector sensors relative to the field during operation of the gradiometer.

2.   The method of claim 1 wherein the at least first and second field vector sensors of the gradiometer are substantially axially aligned, such that the gradiometer is an axial gradiometer.

10   3.   The method of claim 1 or claim 2 wherein the step of controllably altering the position of the at least first and second field vector sensors is performed by rotating the at least first and second field vector sensors about an axis of rotation.

4.   The method of claim 3 wherein the step of controllably altering the

15   position of the at least first and second field vector sensors is performed by rotating the at least first and second field vector sensors continuously during operation of the gradiometer.

5.   The method of claim 4, further comprising the step of providing a motor for driving the gradiometer at a substantially constant revolutional velocity

20   about the axis of rotation.

6.   The method of claim 3 wherein the step of controllably altering the position of the at least first and second field vector sensors is performed by rotating the at least first and second field vectors piecewise about an axis of rotation.

25   7.   The method of claim 3 when dependent on claim 2 or any one of claims 4 to 6 when dependent on claims 2 and 3, wherein the axis of rotation is positioned substantially perpendicular to the axial alignment of the first and second field vector sensors.

8.   The method of claim 7 wherein the axis of rotation is positioned

30   substantially between the first and second field vector sensors.

9.   The method of claim 7 or claim 8, wherein the axis of rotation is positioned substantially equidistant from the first and second field vector sensors.

10.   The method of any one of the preceding claims wherein the gradiometer

35   is an axial magnetic gradiometer.

11.   The method of claim 10 wherein the field vector sensors of the axial magnetic gradiometer are SQUIDs.

12.   The method of claim 10 wherein the field vector sensors of the axial magnetic gradiometer are flux gates.

5   13.   The method of claim 10 wherein the field vector sensors of the axial magnetic gradiometer are superconducting pick-up loops.

14.   The method of any one of claims 10 to 13 wherein the sensitivity vectors of the field vector sensors lie substantially in a nominal x-y plane.

15.   The method of claim 14 wherein the axial magnetic gradiometer is

10   rotated about a nominal z-axis perpendicular to the x-y plane.

16.   The method of any one of claims 10 to 15 further comprising the step of retrieving information relating to the magnetic field from the axial magnetic gradiometer as its position is controllably altered.

17.   The method of claim 16 wherein the information relating to the magnetic

15   field comprises the g_{xy} component and linear combinations of the values of the g_{xx} and g_{yy} components of a field gradient tensor, in addition to information about the B_x and B_y field components.

18.   The method of claim 16 or claim 17 wherein the step of retrieving information relating to the magnetic field further comprises transforming a

20   received signal from the field vector sensors into the Fourier domain in order to retrieve the information relating to the magnetic field.

19.   The method of any one of the preceding claims further comprising the steps of:

        providing second and third gradiometers, each of the gradiometers

25   having at least first and second axially aligned field vector sensors connected in a differencing arrangement;

        positioning the gradiometers such that an axis of each gradiometer is not parallel to an axis of any other of the gradiometers; and

        controllably altering a position of the at least first and second field vector

30   sensors of each gradiometer relative to the field during operation of that gradiometer.

20.   A field detection device comprising:

        a gradiometer, the gradiometer having at least first and second field vector sensors connected in a differencing arrangement; and

35   means for controllably altering the position of the at least first and second field vector sensors relative to a field during operation of the gradiometer.

21. The device of claim 20, wherein the at least first and second field vector sensors of the gradiometer are substantially axially aligned, such that the gradiometer is an axial gradiometer

22. The device of claim 20 or claim 21, wherein the means for controllably
5   altering the position of the at least first and second field vector sensors comprises means for rotating the at least first and second field vector sensors about an axis of rotation.

23. The.device of claim 22 wherein the means for rotating the at least first and second field vector sensors about the axis of rotation is operable to rotate
10   the at least first and second field vector sensors continuously during operation of the gradiometer.

24. The device of claim 23 wherein the means for rotating is a motor operable to drive the gradiometer at a substantially constant revolutional. velocity about the axis of rotation.

15   25. The device of claim 22, wherein the means. for rotating the at least first and second field vector sensors about the axis of rotation is operable to rotate the gradiometer piecewise about the axis of rotation.

26. The device of claim 22 when dependent on claim 21 or any one of claims . 23 to 25 when dependent on claims 21 and 22, wherein the axis of rotation is
20   positioned substantially perpendicular to the co-axial first and second field vectors.

27. The device of claim 26 wherein the axis of rotation is positioned substantially between the first and second field vector sensors.

28. The device of claim 26 or claim 27 wherein the axis of rotation is
25   positioned substantially equidistant from the first and second field vector sensors.

29. The device of any one of claims 20 to 18, wherein the gradiometer is an axial gradiometer.

30. The device of claim 29, wherein the field vector sensors of the magnetic
30   axial gradiometer are SQUIDs.

31. The device of claim 29, wherein the field vector sensors of the magnetic axial gradiometer are flux gates.

32. The device of claim 29, wherein the field vector sensors of the magnetic axial gradiometer are superconducting pick-up loops.

35   33. The device of any one of claims 29 to 32 wherein the sensitivity vectors of the field vector sensors lie substantially in a nominal x-y plane.

34.   The device of claim 33 wherein the means for controllably altering the position of the at least first and second field vector sensors comprises means for rotating the at least first and second field vector sensors about a nominal z-axis perpendicular to the x-y plane.

5   35.   The device of any one of claims 29 to 34 further comprising means for retrieving information relating to the magnetic field from the axial magnetic gradiometer as its position is controllably altered.

36.   The device of claim 35 wherein the information relating to the magnetic field comprises the g_{xy} component and linear combinations of the values of the

10   g_{xx} and g_{yy} components of a field gradient tensor, in addition to information about the B_x and B_y field components.

37.   The device of claim 35 or claim 36 further comprising means for transforming a received signal from the field vector sensors into the Fourier domain in order to retrieve the information relating to the magnetic field.

15   38.   The device of any one of claims 20 to 37, further comprising:

second and third gradiometers, each of the gradiometers having at least first and second field vector sensors connected in a differencing arrangement, the gradiometers being positioned such that an axis of each gradiometer is not parallel to an axis of any other of the gradiometers; and

20   means for controllably altering a position of the at least first and second field vector sensors of each gradiometer relative to the field during operation of that gradiometer.

39.   A method of obtaining a complete magnetic gradient tensor of a magnetic field, the method comprising:

25   providing at least three axial gradiometers in the magnetic field such that an axis of each axial gradiometer is not parallel to an axis of any other one of the at least three gradiometers; and

controllably altering a position of each of the at least three gradiometers relative to the magnetic field during operation of the gradiometer.

30   40.   The method of claim 39 wherein the step of controllably altering comprises rotating each of the at least three axial gradiometers about a respective axis of rotation of that gradiometer.

41.   The method of claim 40 wherein the axis of rotation of each gradiometer is not parallel to an axis of rotation of any other one of the at least three

35   gradiometers.

42.   The method of claim 41 or claim 42 wherein the at least three gradiometers are rotated at a substantially constant velocity about their respective axes of rotation.

43.   The method of claim 42 further comprising the step of transforming obtained data to the Fourier domain.

44.   The method of claim 43 further comprising the step of distinguishing field gradient information from field information in the Fourier domain.

45.   The method of claim 43 or claim 44 further comprising distinguishing information about the g_{xy} component of the gradient tensor from information due to the diagonal gradient components, even at the same frequency.

46.   The method of any one of claims 43 to 45 wherein the Fourier transform is performed by a fast Fourier transform algorithm.

47.   The method of any one of claims 43 to 45 wherein the Fourier transform is performed by a simple discrete implementation via a coarse sampling of a discrete number of data points per cycle of the rotation.

48.   The method of claim 47 wherein eight data points are sampled.

49.   The method of claim 47 or claim 48 wherein the data points are positioned equally around the 360 degree rotation.

50.   The method of any one of claims 42 to 49 further comprising the step of applying a least-squares best-fit solution to obtained data.

51.   The method of any one of claims 42 to 50 wherein DC offsets are determined and monitored to provide information about the operating conditions of the gradiometers.

52.   The method of claim 51 wherein the DC offsets comprise one or both of: low frequency drift in at least one field vector sensor of the at least three gradiometers; and the fixed offset of at least one field vector sensor of the at least three gradiometers.

53.   The method of any one of claims 42 to 52, further comprising the step of isolating the at least three gradiometers from physical vibration occurring in the vicinity of their respective fundamental and first harmonic frequencies of rotation.

54.   The method of any one of claims 42 to 53 wherein the at least three gradiometers are rotated at differing frequencies, in order to facilitate separation of their data signals in the Fourier domain.

55.   A device for obtaining a complete magnetic gradient tensor of a magnetic field, the device comprising:

at least three axial gradiometers positioned such that an axis of each axial gradiometer is not parallel to an axis of any other one of the at least three gradiometers; and

means for controllably altering a position of each of the at least three
5 gradiometers relative to the magnetic field during operation of the gradiometer.

56. The device of claim 55 wherein the means for controllably altering are means for rotating each of the at least three axial gradiometers about a respective axis of rotation of that gradiometer.

57. The device of claim 56 wherein the axis of rotation of each gradiometer
10 is not parallel to an axis of rotation of any other one of the at least three gradiometers.

58. The device of claim 56 or claim 57, wherein the at least three gradiometers are rotated at a substantially constant velocity about their respective axes of rotation.

15 59. The device of claim 58, further comprising means for converting data obtained from the at least three gradiometers to the Fourier domain.

60. The method of claim 58 or claim 59 wherein the at least three gradiometers are rotated at differing frequencies, in order to facilitate separation of their data signals in the Fourier domain.

20 61. The device of any one of claims 56 to 60 further comprising means for applying a least-squares best fit solution to obtained data.

62. The device of any one of claims 56 to 61 further comprising means for detecting and measuring a DC offset.

63. The device of claim 62 wherein the DC offset comprises one or both of:
25 low frequency drift in at least one field vector sensor of the at least three gradiometers; and the fixed offset of at least one field vector sensor of the at least three gradiometers.

64. The device of claim 62 or claim 63 further comprising means for determining operating conditions of one or more of the at least three
30 gradiometers based on said detected and measured DC offset.

65. The device of any one of claims 62 to 64 wherein said means for detecting and measuring a DC offset separates the DC offset from field data and gradient data in the Fourier domain.

66. The device of any one of claims 56 to 65 further comprising means for
35 distinguishing field gradient information from field information in the Fourier domain.

67.	The device of any one of claims 56 to 66 further comprising means for distinguishing information about the g_{xy} component of the gradient tensor from information due to the diagonal gradient components, even at the same frequency.

68.	The device of claim 59 wherein the Fourier transform is implemented by means of a fast Fourier transform algorithm.

69.	The device of claim 59 wherein the Fourier transform is implemented by sampling discrete data points throughout each cycle of rotation.

70.	The device of claim 69 wherein eight discrete data points are sampled.

71.	The device of claim 69 or 70 wherein the discrete data points are equidistant throughout the cycle of rotation.

72.	The device of any one of claims 56 to 71 further comprising means for isolating the at least three gradiometers from physical vibration occurring in the vicinity of their respective fundamental and first harmonic frequencies of rotation.
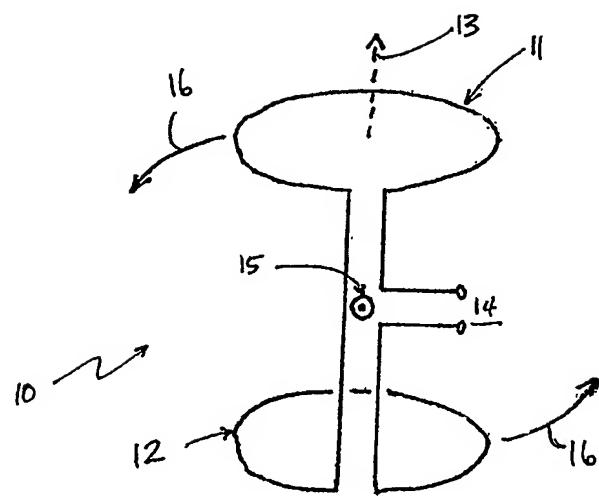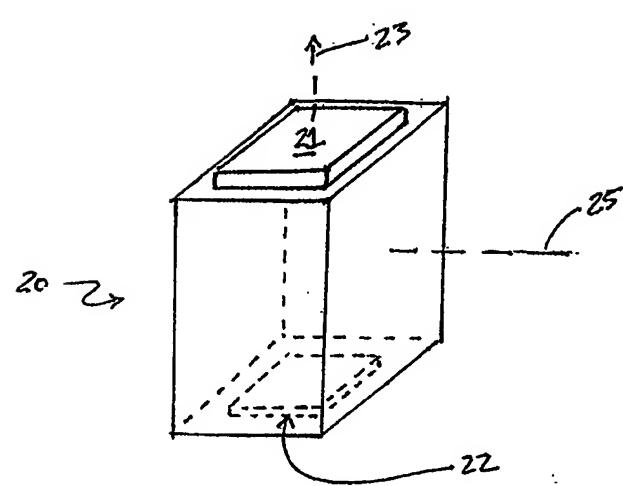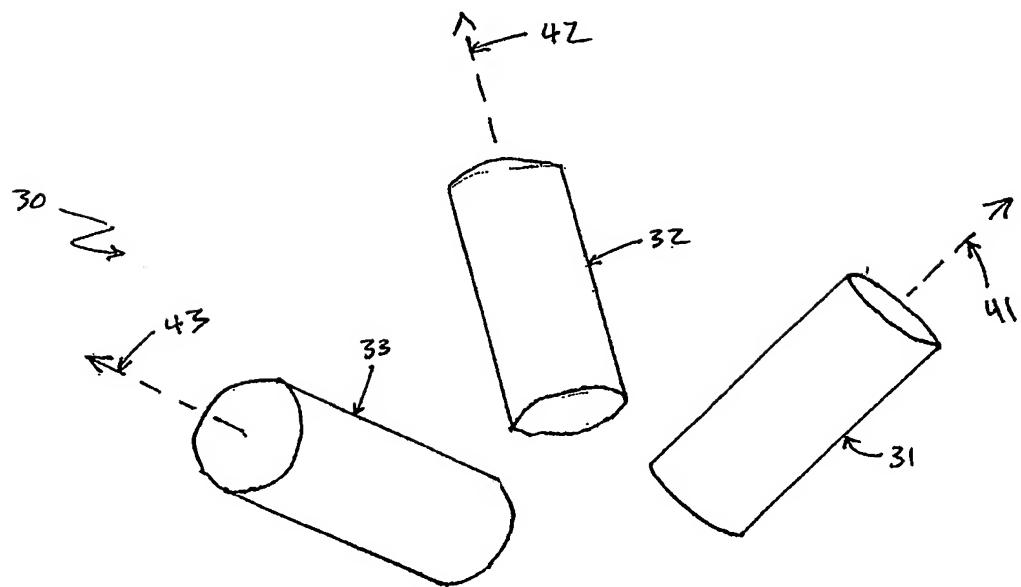
FIGURE IA

FIGURE IB

FIGURE 2

INTERNATIONAL SEARCH REPORT

A.	CLASSIFICATION OF SUBJECT MATTER

Int. Cl. [7]: G01R 33/022, 33/00, 33/035

According to International Patent Classification (IPC) or to both national classification and IPC

B.	FIELDS SEARCHED

Minimum documentation searched (classification system followed by classification symbols)

Documentation searched other than minimum documentation to the extent that such documents are included in the fields searched

Electronic data base consulted during the international search (name of data base and, where practicable, search terms used)
WPAT keywords: gradiometer, vector, altering, moving, displacing, rotating and similar terms

C.	DOCUMENTS CONSIDERED TO BE RELEVANT

Category*	Citation of document, with indication, where appropriate, of the relevant passages	Relevant to claim No.
X Y	US 5357802 A (HOFMEYER et al) 25 October 1994 abstract, column 5 lines 3-34, Fig 3A	1, 2, 20, 21 39, 55
Y	GB 2258314 A (DORNIER GmbH) 3 February 1993 abstract, Fig 2	39, 55
Y	US 5469056 A (ESCHNER et al) 21 November 1995 abstract, Fig 2	39, 55

X	Further documents are listed in the continuation of Box C	X	See patent family annex

* Special categories of cited documents:
"A" document defining the general state of the art which is not considered to be of particular relevance
"E" earlier application or patent but published on or after the international filing date
"L" document which may throw doubts on priority claim(s) or which is cited to establish the publication date of another citation or other special reason (as specified)
"O" document referring to an oral disclosure, use, exhibition or other means
"P" document published prior to the international filing date but later than the priority date claimed

"T" later document published after the international filing date or priority date and not in conflict with the application but cited to understand the principle or theory underlying the invention
"X" document of particular relevance; the claimed invention cannot be considered novel or cannot be considered to involve an inventive step when the document is taken alone
"Y" document of particular relevance; the claimed invention cannot be considered to involve an inventive step when the document is combined with one or more other such documents, such combination being obvious to a person skilled in the art
"&" document member of the same patent family

Date of the actual completion of the international search 24 October 2003	Date of mailing of the international search report 0 3 NOV 2003
Name and mailing address of the ISA/AU AUSTRALIAN PATENT OFFICE PO BOX 200, WODEN ACT 2606, AUSTRALIA E-mail address: pct@ipaustralia.gov.au Facsimile No. (02) 6285 3929	Authorized officer **Mani Ramachandran** Telephone No : (02) 6283 2233

C (Continuation).	DOCUMENTS CONSIDERED TO BE RELEVANT	
Category*	Citation of document, with indication, where appropriate, of the relevant passages	Relevant to claim No.
A, P	David Gleason, Air Force Research Laboratory, Space Vehicle Directorate, Hanscom AFB, MA, presentation at DARPA PINS Meeting, 27 May 2003, "Critical Role of Gravity Compensation in a Stand-Alone Precision INS" http://www.eventmakeronline.com/dso/PDF/AFRL-Gleason.pdf	
A	US 4492923 A (BYRAM) 8 January 1985 whole document	

Note: For the Y indications, US 5357802 can be combined with any one of
GB 2258314 or US 5469056 with relevance to the same claims.

This Annex lists the known "A" publication level patent family members relating to the patent documents cited in the above-mentioned international search report. The Australian Patent Office is in no way liable for these particulars which are merely given for the purpose of information.

Patent Document Cited in Search Report		Patent Family Member							
US	5357802	NONE							
GB	2258314	DE	4005079	US	5311129	US	5469056		
		FR	2658615	FR	2680008				
US	5469056	DE	4005079	FR	2658615	GB	2258314		
		US	5311129	FR	2680008				
US	4492923	NONE							
							END OF ANNEX		